

LINK COBORDISMS AND ABSOLUTE GRADINGS ON LINK FLOER HOMOLOGY

IAN ZEMKE

ABSTRACT. We give an alternate construction of absolute Maslov and Alexander gradings on link Floer homology, using a surgery presentation of the link complement. We show that the link cobordism maps defined by the author are graded and satisfy a grading change formula. We also show that our construction agrees with the original definition of the symmetrized Alexander grading defined by Ozsváth and Szabó. As an application, we compute the link cobordism maps associated to a closed surface in S^4 with a simple dividing set. For simple dividing sets, the maps are determined by the genus of the surface. We show how the grading formula quickly recovers some known bounds on the concordance invariants $\tau(K)$ and $\Upsilon_K(t)$, and use it to prove a new bound for $\Upsilon_K(t)$ for knot cobordisms in negative definite 4-manifolds. As a final application, we recover an adjunction relation for the ordinary Heegaard Floer cobordism maps as a consequence of our grading formula and properties of the link Floer cobordism maps. This gives a link Floer interpretation of adjunction relations and inequalities appearing in Seiberg–Witten and Heegaard Floer theories.

CONTENTS

1. Introduction	2
1.1. Bounds on the concordance invariants $\Upsilon_K(t)$ and $\tau(K)$	2
1.2. Absolute gradings on link Floer homology and general grading change formulas	3
1.3. Adjunction relations from link Floer homology	5
1.4. The maps on \mathcal{HFL}^∞ for surfaces in negative definite 4-manifolds	6
1.5. Further remarks	7
1.6. Organization	7
1.7. Acknowledgments	7
2. Background and definitions	8
2.1. Curved link Floer homology	8
2.2. Colorings of links and surfaces, and link cobordism maps	8
2.3. Grading assignments, J -null-homologous links, and J -graded colorings of links	9
3. Heegaard triples and Spin^c structures	10
3.1. Constructing 4-manifolds and Spin^c structures from Heegaard triples	10
3.2. Heegaard triples and bouquets of framed 1-dimensional links	14
4. Kirby calculus for manifolds with boundary	14
5. Relative gradings	19
5.1. The relative Maslov gradings	19
5.2. The relative Alexander multi-grading	19
5.3. Two simple examples	21
5.4. Absolute gradings on $\mathcal{CFL}^\circ((S^1 \times S^2)^{\#k}, \mathbb{U}, \mathfrak{s}_0)$ for unlinks \mathbb{U}	21
6. Coherent gradings on coherent chain homotopy types	22
6.1. Change of diagrams maps for Alexander gradings	23
6.2. Change of diagrams maps for Maslov gradings	25
6.3. Coherent gradings	26
7. Absolute Alexander multi-gradings	26
7.1. Construction of the absolute grading $A_{Y, \mathbb{L}, S, \mathbb{P}}$	26
7.2. Invariance of the absolute Alexander multi-grading	27
7.3. Gradings using α -bouquets	34
7.4. Dependence on the Seifert surface S	34

8. Absolute Maslov gradings	36
9. Link cobordisms and grading change formulas	37
9.1. Grading changes of maps associated to elementary link cobordisms	37
10. Conjugation symmetry, equivalence with other constructions, and collapsing gradings	43
10.1. Conjugation symmetry	43
10.2. Collapsing gradings	45
11. A new proof of a bound of Ozsváth and Szabó on τ	45
12. t -modified knot Floer homology and a bound on the $\Upsilon_K(t)$ invariant	46
12.1. Additional examples of the bound	49
12.2. Positive torus knots	50
13. The adjunction relation	50
14. The maps on \mathcal{HFL}^∞ for surfaces in negative definite 4-manifolds	55
References	57

1. INTRODUCTION

In [OS04b] and [OS06], Ozsváth and Szabó define a collection of invariants, known as Heegaard Floer homology, for closed, oriented 3-manifolds. To a 3-manifold Y and a Spin^c structure, they construct modules $HF^\circ(Y, \mathfrak{s})$ for $\circ \in \{-, +, \infty, \wedge\}$. If $W : Y_1 \rightarrow Y_2$ is a cobordism and $\mathfrak{s} \in \text{Spin}^c(W)$ is a Spin^c structure, Ozsváth and Szabó define cobordism maps from $HF^\circ(Y_1, \mathfrak{s}|_{Y_1})$ to $HF^\circ(Y_2, \mathfrak{s}|_{Y_2})$. An important property of the cobordism maps is that they are graded with respect to an absolute grading, and induce grading change

$$\frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

The cobordism maps have particularly nice properties with respect to negative definite 4-manifolds, and the previously mentioned grading change formula gives, for example, a proof of Donaldson's theorem on the intersection forms of negative definite 4-manifolds (see [OS03a]).

There is a refinement of Heegaard Floer homology for knots embedded in a 3-manifold, discovered by Ozsváth and Szabó in [OS04d] and independently by Rasmussen in [Ras03]. Ozsváth and Szabó also define a refinement for links in [OS08]. In [Zem16b], the author describes a collection of cobordism maps forming a “link Floer TQFT”, which are defined on a version of the full link Floer complex. In this paper, we consider the grading changes associated to the maps constructed in [Zem16b], and consider also some consequences for several knot invariants defined using knot Floer homology.

1.1. Bounds on the concordance invariants $\Upsilon_K(t)$ and $\tau(K)$. In [OS03b], using knot Floer homology Ozsváth and Szabó construct a homomorphism τ from the smooth concordance group to \mathbb{Z} . It follows from the bound they prove on τ that if $(W, \Sigma) : (S^3, K_1) \rightarrow (S^3, K_2)$ is an oriented knot cobordism with $b_1(W) = b_2^+(W) = 0$, then

$$(1) \quad \tau(K_2) \leq \tau(K_1) - \frac{||[\Sigma]|| + [\Sigma] \cdot [\Sigma]}{2} + g(\Sigma)$$

where $||[\Sigma]||$ is the “ L^1 norm” of $[\Sigma] \in H_2(W; \mathbb{Z})$. In Section 11, we will give a new proof of this result, and see how it follows quickly from a grading change formula for the link cobordism maps from [Zem16b] which we prove in this paper. We state the more general grading change formulas in the next section of the introduction.

Using the previously mentioned grading change formulas for the Alexander and Maslov gradings, we prove a similar bound on the concordance invariant $\Upsilon_K(t)$, defined in [OSS14]. The invariant $\Upsilon_K(t)$ is a piecewise linear function from $[0, 2]$ to \mathbb{R} . It determines a homomorphism from the concordance group to the group of piecewise linear functions from $[0, 2]$ to \mathbb{R} .

If s is an integer and $t \in [0, 2]$ is a real number, we define the quantity

$$M_t(s) = \max_{a \in 2\mathbb{Z}+1} \frac{-a^2 + 1 + 2ast - 2s^2t}{4}.$$

If W is a negative definite 4-manifold with boundary equal to two copies of S^3 , we pick an orthonormal basis e_1, \dots, e_n of $H_2(W; \mathbb{Z})$. If $[\Sigma] \in H_2(W; \mathbb{Z})$ is a homology class, we write $[\Sigma] = s_1 \cdot e_1 + \dots + s_n \cdot e_n$ for integers s_i and define

$$M_t([\Sigma]) = \sum_{i=1}^n M_t(s_i),$$

a piecewise linear function over $t \in [0, 2]$. Note that $M_t([\Sigma])$ also has the more invariant description

$$M_t([\Sigma]) = \max_{C \in \text{Char}(W)} \frac{C^2 + b_2(W) - 2t\langle C, [\Sigma] \rangle + 2t([\Sigma] \cdot [\Sigma])}{4},$$

where $\text{Char}(W)$ is the set of characteristic vectors of $H^2(W; \mathbb{Z})$.

We prove the following bound on $\Upsilon_K(t)$:

Theorem 1.1. *If $(W, \Sigma) : (S^3, K_1) \rightarrow (S^3, K_2)$ is an oriented knot cobordism with $b_1(W) = b_2^+(W) = 0$, then*

$$\Upsilon_{K_2}(t) \geq \Upsilon_{K_1}(t) + M_t([\Sigma]) + g(\Sigma) \cdot (|t - 1| - 1).$$

For small t , one has $\Upsilon_K(t) = -\tau(K) \cdot t$, and correspondingly for small t our bound reads

$$\Upsilon_{K_2}(t) \geq \Upsilon_{K_1}(t) + t \cdot \left(\frac{|[\Sigma]| + [\Sigma] \cdot [\Sigma]}{2} - g(\Sigma) \right),$$

reflecting the Ozsváth–Szabó bound on τ in Equation (1).

Interestingly, using the computation of $\Upsilon_{T_{n,n+1}}(t)$ in [OSS14] for the $(n, n+1)$ -torus knot, we note that in fact

$$M_t(n) = \Upsilon_{T_{n,n+1}}(t).$$

In more generality, the $\Upsilon_K(t)$ invariant is computed for torus knots by Feller and Krcatovich in [FK16]. Comparing our bound with their computation shows that our bound is sharp, when applied to naturally appearing knot cobordisms between positive torus knots.

In Section 12 we consider some further applications and examples of the above bound. We give several alternate proofs and extensions of results from [OSS14]. For example if $(W, \Sigma) : (S^3, K_1) \rightarrow (S^3, K_2)$ is an oriented knot cobordism such that W is a rational homology cobordism, the above theorem immediately yields

$$|\Upsilon_{K_2}(t) - \Upsilon_{K_1}(t)| \leq t \cdot g(\Sigma).$$

As another example we give a new proof of the bound from [OSS14] for $\Upsilon_K(t)$ after crossing changes.

1.2. Absolute gradings on link Floer homology and general grading change formulas. The previous theorem follows from a much more general result about the grading changes associated to the link cobordism maps from [Zem16b]. We now outline our results about grading changes on link Floer homology.

Definition 1.2. An **oriented multi-based link** $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ in Y^3 is an oriented link L with two disjoint collections of basepoints $\mathbf{w} = \{w_1, \dots, w_n\}$ and $\mathbf{z} = \{z_1, \dots, z_n\}$, such that as one traverses the link the basepoints alternate between \mathbf{w} and \mathbf{z} . Also, each component of L has at least two basepoints, and each component of Y contains a component of L .

To a multi-based link \mathbb{L} in Y^3 equipped with a Spin^c structure $\mathfrak{s} \in \text{Spin}^c(Y)$, we can construct a “curved” chain complex

$$\mathcal{CFL}^\circ(Y, \mathbb{L}, \mathfrak{s})$$

over the ring $\mathbb{Z}_2[U_{\mathbf{w}}, V_{\mathbf{z}}]$ which is filtered by $\mathbb{Z}^{\mathbf{w}} \oplus \mathbb{Z}^{\mathbf{z}}$. Here $\mathbb{Z}_2[U_{\mathbf{w}}, V_{\mathbf{z}}]$ denotes the polynomial ring generated by the variables U_w and V_z for $w \in \mathbf{w}$ and $z \in \mathbf{z}$. A curved chain complex is a module with an endomorphism ∂ squaring to the action of a scalar in the ground ring (sometimes these are called matrix factorizations).

Suppose that J is an index set and L is a link. We call a map $\pi : C(L) \rightarrow J$, from the set of components of L to J , a **grading assignment**. We say that L is **J -null-homologous** if

$$[\pi^{-1}(j)] = 0 \in H_1(Y; \mathbb{Z})$$

for all $j \in J$. We define a J -**Seifert surface** of L to be a collection of immersed surfaces S_j (indexed over $j \in J$) such that each S_j has oriented boundary equal to the sublink $-\pi^{-1}(j) \subseteq L$. The components S_j may intersect for various j .

In [OS08], Ozsváth and Szabó, one can define a distinguished Alexander grading on $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$ for a link \mathbb{L} in an integer homology sphere Y . However it isn't immediately obvious how to compute the grading change induced by the link cobordism maps from [Zem16b] using their construction. Using surgery presentations of link complements, we provide a description which is more natural from the link cobordism perspective:

Theorem 1.3. (a) *If $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ is a multi-based link in Y , with a grading assignment $\pi : C(L) \rightarrow J$ such that L is J -null-homologous, and S is a choice of J -Seifert surface, then the modules $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$ possess a distinguished absolute Alexander multi-grading $A_{Y, \mathbb{L}, S}$ over \mathbb{Q}^J .*

(b) *If \mathfrak{s} is torsion, the multi-grading $A_{Y, \mathbb{L}, S}$ is independent of the J -Seifert surface S . More generally if S and S' are two choices of J -Seifert surfaces, then*

$$A_{Y, \mathbb{L}, S'}(\mathbf{x})_j - A_{Y, \mathbb{L}, S}(\mathbf{x})_j = \frac{\langle c_1(\mathfrak{s}), [S'_j \cup -S_j] \rangle}{2}.$$

(c) *Under only the assumption that \mathfrak{s} is torsion, there is an absolute Maslov grading $\text{gr}_{\mathbf{w}}$ on $CFL^\infty(Y, \mathbb{L}, \mathfrak{s})$.*

Under only the assumption that $\mathfrak{s} - PD[L]$ is torsion, there is an absolute Maslov grading $\text{gr}_{\mathbf{z}}$.

(d) *If $[L] = 0 \in H_1(Y; \mathbb{Z})$ and \mathfrak{s} is torsion, then all gradings are defined, and $A = \frac{1}{2}(\text{gr}_{\mathbf{w}} - \text{gr}_{\mathbf{z}})$, where A denotes the collapsed Alexander grading.*

The $\text{gr}_{\mathbf{w}}$ -grading is obtained by paying attention to only the \mathbf{w} -basepoints, while the $\text{gr}_{\mathbf{z}}$ -grading is obtained by paying attention to only the \mathbf{z} -basepoints. For a J -null-homologous link L , the j -component, $(A_{Y, \mathbb{L}, \mathfrak{s}})_j$, of the multi-grading takes values in $\mathbb{Z} + \frac{1}{2}\ell k(L \setminus L_j, L_j)$ (see Remark 7.1).

The following table shows the grading changes associated to the action of the variables for the various gradings:

Variable	A_j -grading change	$\text{gr}_{\mathbf{w}}$ -grading change	$\text{gr}_{\mathbf{z}}$ -grading change
U_w	$-\delta(\pi(w), j)$	-2	0
V_z	$+\delta(\pi(z), j)$	0	-2

Here $\delta(\pi(w), j)$ is 1 if w is on a link component assigned grading j by π , and zero otherwise.

We show in Section 10 that our construction of the absolute Alexander grading has a symmetry with respect to conjugation which also characterizes the absolute grading defined in [OS08] (for links in integer homology spheres with two basepoints per component). In particular, our construction of the absolute grading agrees with the one due to Ozsváth and Szabó for links in those cases.

As our grading agrees with the one defined by Ozsváth and Szabó in the most interesting cases, the above theorem is not of great interest in itself. Nonetheless, our description of the gradings has the advantage that we can compute the grading change associated to the maps in [Zem16b] induced by link cobordisms. The maps from [Zem16b] use the following notion of morphism between multi-based links in 3-manifolds, originally from [Juh16]:

Definition 1.4. We say a pair (W, F) is a **decorated link cobordism** between two 3-manifolds with multi-based links, and write

$$(W, F) : (Y_1, (L_1, \mathbf{w}_1, \mathbf{z}_1)) \rightarrow (Y_2, (L_2, \mathbf{w}_2, \mathbf{z}_2)),$$

if

- (1) W is a 4-dimensional cobordism from Y_1 to Y_2 ;
- (2) $F = (\Sigma, \mathcal{A})$ consists of an oriented surface Σ with a properly embedded 1-manifold \mathcal{A} dividing Σ into two disjoint subsurfaces $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$, meeting along \mathcal{A} ;
- (3) $\partial\Sigma = -L_1 \sqcup L_2$
- (4) Each component of $L_i \setminus \mathcal{A}$ contains exactly one basepoint;
- (5) The \mathbf{w} -basepoints are all in $\Sigma_{\mathbf{w}}$ and the \mathbf{z} -basepoints are all in $\Sigma_{\mathbf{z}}$.

In [Juh16], Juhász gave a construction of link cobordism maps for \widehat{HFL} , the hat flavor of link Floer homology, for decorated link cobordisms in the above sense. The construction used the contact gluing map

from [HKM08] as well as handle attachment maps similar to the ones from [OS06]. To such a decorated link cobordism, in [Zem16b] the author constructed maps

$$F_{W,F,\mathfrak{s}}^\circ : CFL^\circ(Y_1, \mathbb{L}_1, \mathfrak{s}|_{Y_1}) \rightarrow CFL^\circ(Y_2, \mathbb{L}_2, \mathfrak{s}|_{Y_2}),$$

which are invariants up to filtered, equivariant chain homotopy. These maps are expected to agree with the maps defined by Juhász when restricted to the hat flavor.

To define functorial maps, we actually need an extra piece of data: a “coloring”. A coloring is an assignment of formal variables to the regions of $\Sigma \setminus \mathcal{A}$. Different colorings are useful for different applications. After coloring, the maps become homomorphisms over the ring generated by our collection of formal variables. In Section 2.3, we define a notion of a J -graded coloring, which is a coloring that is compatible with a grading assignment over J , and allows for the maps and chain complexes to be graded.

We prove the following grading change formula for the Alexander grading:

Theorem 1.5. *Suppose that $(W, F) : (Y_1, \mathbb{L}_1) \rightarrow (Y_2, \mathbb{L}_2)$ is a decorated link cobordism with $F = (\Sigma, \mathcal{A})$ and $\pi : C(\Sigma) \rightarrow J$ is a grading assignment, such that the links on the ends are J -null-homologous with respect to the induced grading assignment. Suppose further that S_1 and S_2 are J -Seifert surfaces for \mathbb{L}_1 and \mathbb{L}_2 , respectively. For a J -graded coloring of F , the map $F_{W,F,\mathfrak{s}}^\circ$ is graded with respect to the absolute Alexander multi-grading over J . For $j \in J$ the map satisfies*

$$A_{Y_2, \mathbb{L}_2, S_2}(F_{W,F,\mathfrak{s}}^\circ(\mathbf{x}))_j - A_{Y_1, \mathbb{L}_1, S_1}(\mathbf{x})_j = \frac{\langle c_1(\mathfrak{s}), \widehat{\Sigma}_j \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2} + \frac{\chi(\Sigma_{\mathbf{w},j}) - \chi(\Sigma_{\mathbf{z},j})}{2},$$

for a homogeneous \mathbf{x} . Here $\Sigma_j = \pi^{-1}(j)$, and $\widehat{\Sigma}_j$ is the surface obtained by capping Σ_j with $-(S_1)_j$ and $(S_2)_j$. Similarly $\Sigma_{j,\mathbf{w}}$ and $\Sigma_{j,\mathbf{z}}$ denote the type- \mathbf{w} and $-\mathbf{z}$ regions of $\Sigma_j \setminus \mathcal{A}$.

We also prove grading change formulas for the $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ gradings:

Theorem 1.6. *Suppose that $(W, F) : (Y_1, \mathbb{L}_1) \rightarrow (Y_2, \mathbb{L}_2)$ is a decorated link cobordism with a J -graded coloring. If \mathfrak{s} is torsion on the ends of W , then the maps $F_{W,F,\mathfrak{s}}^\circ$ are graded with respect to $\text{gr}_{\mathbf{w}}$ and satisfy*

$$\text{gr}_{\mathbf{w}}(F_{W,F,\mathfrak{s}}^\circ(\mathbf{x})) - \text{gr}_{\mathbf{w}}(\mathbf{x}) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4} + \widetilde{\chi}(\Sigma_{\mathbf{w}}),$$

for homogeneously graded \mathbf{x} . Similarly, if $\mathfrak{s} - PD[\Sigma]$ is torsion on the ends of W , then the maps $F_{W,F,\mathfrak{s}}^\circ$ are graded with respect to $\text{gr}_{\mathbf{z}}$ and satisfy

$$\text{gr}_{\mathbf{z}}(F_{W,F,\mathfrak{s}}^\circ(\mathbf{x})) - \text{gr}_{\mathbf{z}}(\mathbf{x}) = \frac{c_1(\mathfrak{s} - PD[\Sigma])^2 - 2\chi(W) - 3\sigma(W)}{4} + \widetilde{\chi}(\Sigma_{\mathbf{z}}),$$

for homogeneous \mathbf{x} .

In the above, the quantity $\widetilde{\chi}(\Sigma_{\mathbf{w}})$ is defined to be

$$\widetilde{\chi}(\Sigma_{\mathbf{w}}) = \chi(\Sigma_{\mathbf{w}}) - \frac{1}{2}(|\mathbf{w}_{\text{in}}| + |\mathbf{w}_{\text{out}}|).$$

The term $\widetilde{\chi}(\Sigma_{\mathbf{z}})$ is defined analogously.

1.3. Adjunction relations from link Floer homology. In a different direction, we consider the maps on the ordinary versions of Heegaard Floer homology obtained from the link cobordism maps by ignoring one set of the basepoints. When we do this, several formulas naturally appear which also appear in Seiberg–Witten theory. Using our grading formula, and some basic properties of the maps from [Zem16b], we prove the following adjunction relation for the ordinary cobordism maps in Heegaard Floer homology:

Theorem 1.7. *Suppose that Σ is a connected, closed, oriented, embedded surface in a cobordism $W : Y_1 \rightarrow Y_2$ with W, Y_1 and Y_2 connected, and that $\mathfrak{s} \in \text{Spin}^c(W)$ is a Spin^c structure with*

$$\langle c_1(\mathfrak{s}), [\Sigma] \rangle - [\Sigma] \cdot [\Sigma] = -2g(\Sigma).$$

Then there is an element $\xi(\Sigma) \in \mathbb{Z}_2[U] \otimes \Lambda^(H_1(\Sigma; \mathbb{Z}))$, such that*

$$F_{W,\mathfrak{s}}^\circ(\cdot) = F_{W,\mathfrak{s}-PD[\Sigma]}^\circ(\iota_*(\xi(\Sigma)) \otimes \cdot),$$

as maps from $(\mathbb{Z}_2[U] \otimes \Lambda^(H_1(W; \mathbb{Z})/\text{Tors})) \otimes HF^\circ(Y_1, \mathfrak{s}|_{Y_1})$ to $HF^\circ(Y_2, \mathfrak{s}|_{Y_2})$ for $\circ \in \{-, +, \infty\}$ and $\iota_* : H_1(\Sigma; \mathbb{Z}) \rightarrow H_1(W; \mathbb{Z})$ is the map induced by inclusion. In fact, we can take $\xi(\Sigma) = \prod_{i=1}^{g(\Sigma)} (U + A_i \cdot B_i)$ for*

any symplectic basis $A_1, \dots, A_n, B_1, \dots, B_n$ of $H_1(\Sigma; \mathbb{Z})$, arising from a collection of simple closed curves A_i and B_i with geometric intersection number $|A_i \cap B_j| = \delta_{ij}$.

Note that we don't require that $g(\Sigma) > 0$. The above should be compared to [OS04c, Theorem 3.1] (for $g(\Sigma) > 0$ and Heegaard Floer), [OS00a, Theorem 1.3] (for $g(\Sigma) > 0$ and Seiberg-Witten) and [FS95, Lemma 5.2] (for $g(\Sigma) = 0$ and Seiberg-Witten).

By applying the previous adjunction relation to the identity cobordism $Y \times [0, 1]$, we get an alternate, link Floer theoretic proof of the standard, 3-dimensional adjunction inequality for Heegaard Floer homology from [OS04a], i.e. if $\Sigma \subseteq Y$ is an embedded, oriented surface of genus $g(\Sigma) > 0$ and $HF^+(Y, \mathfrak{s}) \neq 0$, then

$$|\langle c_1(\mathfrak{s}), [\Sigma] \rangle| \leq 2g(\Sigma) - 2.$$

1.4. The maps on \mathcal{HFL}^∞ for surfaces in negative definite 4-manifolds. Applying the techniques developed in the previous section, we compute the maps on homology associated to knot cobordisms where the underlying 4-manifold is a negative definite and the dividing set is relatively simple.

Theorem 1.8. *Suppose that $(W, F) : (S^3, \mathbb{K}_1) \rightarrow (S^3, \mathbb{K}_2)$ is a knot cobordism such that $b_1(W) = b_2^+(W) = 0$, $F = (\Sigma, \mathcal{A})$ is a surface with divides such that Σ is connected, and \mathbb{K}_i are two knots, each with two basepoints. Suppose further that \mathcal{A} consists of two arcs going from \mathbb{K}_1 to \mathbb{K}_2 and $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$ are both connected. Then the induced map on homology*

$$F_{W, F, \mathfrak{s}}^\infty : \mathcal{HFL}^\infty(Y, \mathbb{K}_1) \rightarrow \mathcal{HFL}^\infty(S^3, \mathbb{K}_2)$$

is an isomorphism. In fact, under the identification $\mathcal{HFL}^\infty(S^3, \mathbb{K}_i) \cong \mathbb{Z}_2[U, V, U^{-1}, V^{-1}]$, it is the map

$$1 \mapsto U^{-d_1/2} V^{-d_2/2},$$

where

$$d_1 = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4} - 2g(\Sigma_{\mathbf{w}})$$

and

$$d_2 = \frac{c_1(\mathfrak{s} - PD[\Sigma])^2 - 2\chi(W) - 3\sigma(W)}{4} - 2g(\Sigma_{\mathbf{z}}).$$

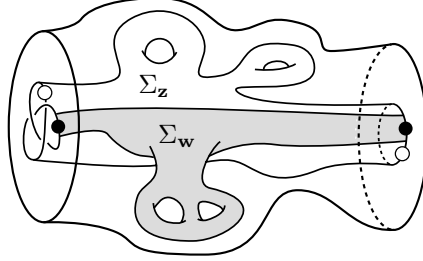


FIGURE 1.1. An example of the dividing sets considered in Theorem 1.8 and Corollary 1.9. Here \mathcal{A} consists of two arcs, and $\Sigma \setminus \mathcal{A}$ consists of two connected components, $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$.

More generally, we can compute explicitly the maps $F_{W, F, \mathfrak{s}}^\infty$ on the homology group \mathcal{HFL}^∞ in terms of the maps $F_{W, \mathfrak{s}}^\infty$ on HF^∞ when the dividing set is as in the previous theorem, even when we don't have $b_1(W) = b_2^+(W) = 0$, though they don't take quite as simple of a form and we can't guarantee they are isomorphisms (see Remark 14.1).

As a consequence of the above theorem, we can compute the maps associated to closed surfaces in S^4 .

Corollary 1.9. *Suppose that $\Sigma \subseteq S^4$ is a closed, oriented surface, and \mathcal{A} is a simple closed curve on Σ which divides Σ into two connected subsurfaces, $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$, then the link cobordism map*

$$F_{S^4, F, \mathfrak{s}_0}^- : CFL^-(\emptyset, \emptyset) \rightarrow CFL^-(\emptyset, \emptyset)$$

is the map

$$1 \mapsto U^{g(\Sigma_{\mathbf{w}})} V^{g(\Sigma_{\mathbf{z}})}.$$

We remark that $CFL^-(\emptyset, \emptyset)$ is identified with $\mathbb{Z}_2[U, V]$. The cobordism map $F_{S^4, F, \mathfrak{s}_0}^-$ is obtained by puncturing S^4 at two points along the arc \mathcal{A} (i.e. removing two balls and computing the map induced by the knot cobordism from (S^3, \mathbb{U}) to (S^3, \mathbb{U})).

1.5. Further remarks. In [JM16b] Juhász and Marengon compute the cobordism maps from [Juh16] on \widehat{HFL} associated to link cobordisms in $S^3 \times [0, 1]$ and in doing so, compute the grading change for such link cobordism maps. For link cobordisms in $S^3 \times [0, 1]$, all terms in our grading formula except the various Euler characteristics of subsets of Σ vanish, and their formula for the change in the Alexander grading and Maslov gradings agrees with ours. See also [JM16a], where their maps are analyzed for concordances.

In the literature, it is more common to use a slightly different version of the full link Floer complex than the “curved link Floer complex” we are using. If $\mathbb{K} = (K, w, z)$ is a doubly based knot in an integer homology sphere Y , one normally considers a $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain complex $CFK^\infty(Y, K)$ over $\mathbb{Z}_2[U, U^{-1}]$. Under our construction of the Alexander grading, we have that

$$CFK^\infty(Y, K) = CFL^\infty(Y, \mathbb{K}, \mathfrak{s}_0)_0,$$

where $CFL^\infty(Y, \mathbb{K}, \mathfrak{s}_0)_0$ denotes the homogeneous subset of zero Alexander grading. Indeed, decomposing over Alexander gradings, we can write

$$CFL^\infty(Y, \mathbb{K}, \mathfrak{s}_0) = \bigoplus_{k \in \mathbb{Z}} CFK^\infty(Y, K)[k].$$

The action of U on $CFK^\infty(Y, K)$ corresponds to the action of $U_w V_z$ on $CFL^\infty(Y, \mathbb{K}, \mathfrak{s}_0)$. Noting that from Theorem 1.3 that $A = \frac{1}{2}(\text{gr}_{\mathbf{w}} - \text{gr}_{\mathbf{z}})$, it follows that the two Maslov gradings $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ coincide on $CFL^\infty(Y, \mathbb{K}, \mathfrak{s}_0)_0 = CFK^\infty(Y, K)$, reflecting the usual convention that CFK^∞ has a single Maslov grading.

We note that in general, if (W, F) is a decorated link cobordism, the cobordism map $F_{W, F, \mathfrak{s}}^\infty$ may not send $CFK^\infty(S^3, K_1)$ (the standard complex) to $CFK^\infty(S^3, K_2)$, since the Alexander grading change may be nonzero. Instead, the link cobordism maps will send $CFK^\infty(S^3, K_1)$ to $CFK^\infty(S^3, K_2)[k]$ for some shift $k \in \mathbb{Z}$ in the Alexander grading (i.e. the cobordism map sends monomials $\mathbf{x} \cdot U^i V^j$ with $A(\mathbf{x}) + (j - i) = 0$ to sums of monomials of the form $\mathbf{y} \cdot U^{i'} V^{j'}$ with $A(\mathbf{y}) + (j' - i') = k$). As such, it is often more convenient to work with the curved link Floer complexes.

Finally, we warn the reader that our normalization conventions for the gradings is somewhat different than is common in the literature, in the case that there are more than two basepoints per link component. We normalize so that if $\mathbb{U} = (U, \mathbf{w}, \mathbf{z})$ is an unlink in S^3 (with possibly many basepoints), the ranks of $\widehat{HFL}(S^3, \mathbb{U})$ in each Maslov grading are symmetric about zero, and similarly the ranks of each Alexander grading are symmetric about zero. Sometimes authors take the convention that $\widehat{HFL}(S^3, \mathbb{U})$ has top degree generator in degree zero for any number of basepoints per link component, so that Heegaard Floer homology is invariant under adding basepoints to a link component. This is less natural from the perspective of link cobordisms. Our convention ensures for example that cobordisms for connected sums like the one from [HMZ] are zero graded.

1.6. Organization. In Sections 2 and 3 we provide some background on link Floer homology, link cobordism maps, and also prove a useful Spin^c structure formula for doubly pointed Heegaard triples. In Section 4 we describe a set of Kirby moves for describing link complements as surgery on a standard unlink complement in S^3 . In Sections 5 and 6 we describe some preliminaries about the relative gradings. In Sections 7 and 8 we define the absolute gradings and prove invariance from the choices in the construction. In Section 9 and 10 we prove the grading change formula for link cobordisms, and some basic properties, such as conjugation invariance and the equivalence with other constructions. In Sections 11 and 12 we prove bounds on $\tau(K)$ and $\Upsilon_K(t)$ using our grading formulas. In Section 13 we see how the adjunction inequality follows from Alexander grading and link cobordism considerations. In Section 14 we compute the link cobordism maps on \mathcal{HFL}^∞ for many negative definite knot cobordisms.

1.7. Acknowledgments. I would like to thank Kristen Hendricks, Jen Hom, András Juhász, David Kravtovich, Robert Lipshitz, Ciprian Manolescu and Marco Marengon for helpful conversations and suggestions.

2. BACKGROUND AND DEFINITIONS

In this section we describe background material about the complexes $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$, and some preliminary definitions which will be used throughout the paper.

2.1. Curved link Floer homology. Knot Floer homology was originally constructed by Ozsváth and Szabó in [OS04d], and independently by Rasmussen in [Ras03]. Link Floer homology is a generalization to links constructed by Ozsváth and Szabó in [OS08]. The link cobordism maps from [Zem16b] are defined on a variation of the construction from [OS08]. We recall some definitions and preliminary results from [Zem16a] and [Zem16b].

To an oriented multi-pointed link \mathbb{L} in Y , we can construct a multi-pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$. We consider the two tori \mathbb{T}_α and \mathbb{T}_β in $\text{Sym}^n(\Sigma)$ defined by

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_n, \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_n$$

where $n = |\alpha| = |\beta|$. Ozsváth and Szabó define a map $\mathfrak{s}_\mathbf{w} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow \text{Spin}^c(Y)$ in [OS04b]. For a choice of $\mathfrak{s} \in \text{Spin}^c(Y)$ and under appropriate admissibility assumptions, we consider the $\mathbb{Z}_2[U_\mathbf{w}, V_\mathbf{z}]$ -module generated by intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}_\mathbf{w}(\mathbf{x}) = \mathfrak{s}$, which we denote by $CFL^-(\mathcal{H}, \mathfrak{s})$. If we choose an almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$, then by counting pseudo-holomorphic curves we can construct an endomorphism ∂ , depending on the almost complex structure. Because of nontrivial Maslov index 2 boundary degenerations bubbling off, the map ∂ doesn't square to zero for a general link. Instead we have that

$$\partial^2(\mathbf{x}) = \sum_{K \in C(L)} (U_{w_{K,1}} V_{z_{K,1}} + V_{z_{K,1}} U_{w_{K,2}} + U_{w_{K,2}} V_{z_{K,2}} + \cdots V_{z_{K,n_K}} U_{w_{K,1}}) \cdot \mathbf{x},$$

where $w_{K,1}, z_{K,1}, \dots, w_{K,n_K}, z_{K,n_K}$ are the basepoints on the link component $K \in C(L)$, in the order that they appear on K (see [Zem16a, Lemma 2.1]). As such, we call CFL° a “curved” chain complex. If each link component has exactly two basepoints, then $\partial^2 = 0$. Also, if we only have two colors (one for the U -variables and one for the V -variables), then $\partial^2 = 0$ as well. The modules $CFL^-(\mathcal{H}, \mathfrak{s})$ also have a natural filtration of $\mathbb{Z}^\mathbf{w} \oplus \mathbb{Z}^\mathbf{z}$, which is given by powers of the variables. The modules $CFL^\infty(\mathcal{H}, \mathfrak{s})$ are defined by allowing all powers of the variables. The more standard complexes $\widehat{CFL}(\mathcal{H}, \mathfrak{s})$ are defined by setting all the variables equal to zero.

There are many diagrams \mathcal{H} for a link. If \mathcal{H} and \mathcal{H}' differ by an elementary Heegaard move, then using the constructions of Ozsváth and Szabó, we can associate a filtered, equivariant chain map $\Phi_{\mathcal{H} \rightarrow \mathcal{H}'} : CFL^\circ(\mathcal{H}) \rightarrow CFL^\circ(\mathcal{H}')$. To a general pair of diagrams \mathcal{H} and \mathcal{H}' , we can define a map $\Phi_{\mathcal{H} \rightarrow \mathcal{H}'}$ as a composition of maps associated to elementary Heegaard moves. The main result of [JT12] implies that the composition is a chain homotopy equivalence and is independent of the choice of intermediate diagrams, up to filtered, equivariant chain homotopy. We define $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$ to be the collection of all of the complexes $CFL^\circ(\mathcal{H}, \mathfrak{s})$, together with the change of diagrams maps. We call the object $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$ the **coherent, equivariant chain homotopy type**. In [Zem16b], we described the natural notion of a morphism between two coherent chain complexes (essentially as a collection of morphisms of chain complexes, which commute up to filtered equivariant chain homotopy with the change of diagrams maps).

2.2. Colorings of links and surfaces, and link cobordism maps. To define link cobordism maps as in [Zem16b], we must algebraically modify the complexes described in the previous section. Since the module $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$ is a module over $\mathbb{Z}_2[U_\mathbf{w}, V_\mathbf{z}]$, a ring which depends on $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$, it doesn't make sense to specify a module homomorphism between the complexes associated to two different links. This ensures that the link cobordism maps are homomorphisms of a fixed module. Also, at many points of the construction in [Zem16b], in order for the maps to be chain maps or to be well defined up to chain homotopy equivalence, one needs to identify certain variables on the complexes. The decoration of the surface with divides encoded exactly which variables needed to be identified to get well defined maps which are chain maps.

To handle these issues, we make the following two definitions:

Definition 2.1. A **coloring of a multi-based link** $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ is a pair (σ, \mathfrak{P}) where \mathfrak{P} is a finite set, indexing a collection of formal variables, and $\sigma : \mathbf{w} \cup \mathbf{z} \rightarrow \mathfrak{P}$ is a map.

Analogously, we make the following definition:

Definition 2.2. A **coloring of a surface with divides** (Σ, \mathcal{A}) is a pair (σ, \mathfrak{P}) where \mathfrak{P} is a finite set, indexing a collection of formal variables, and $\sigma : C(\Sigma_{\mathbf{w}}) \cup C(\Sigma_{\mathbf{z}}) \rightarrow \mathfrak{P}$ is a map from the set of components of $\Sigma \setminus \mathcal{A}$ to \mathfrak{P} .

If (σ, \mathfrak{P}) is a coloring of the multi-based link $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ in Y and $\circ \in \{-, \infty\}$, we can form the colored complex

$$CFL^\circ(Y, \mathbb{L}, \sigma, \mathfrak{P}, \mathfrak{s})$$

by tensoring $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$ with a “coloring module” (see [Zem16a, Section 2]), to formally identify the variable $U_w \in \mathbb{Z}_2[U_{\mathbf{w}}, V_{\mathbf{z}}]$ with $U_{\sigma(w)} \in \mathbb{Z}_2[U_{\mathfrak{P}}]$, and similarly for the V_z variables.

It is important to note that different choices of colorings will be useful for different applications. In the next section, we describe conditions on colorings which will allow us to define Alexander and Maslov gradings.

If (W, F) is a decorated link cobordism and F is given a coloring (σ, \mathfrak{P}) , then the construction of [Zem16b] yields a chain map

$$F_{W, F, \mathfrak{s}}^\circ : CFL^\circ(Y_1, \mathbb{L}_1, \sigma|_{Y_1}, \mathfrak{P}, \mathfrak{s}|_{Y_1}) \rightarrow CFL^\circ(Y_2, \mathbb{L}_2, \sigma|_{Y_2}, \mathfrak{P}, \mathfrak{s}|_{Y_2}).$$

We usually suppress the coloring from the notation.

2.3. Grading assignments, J -null-homologous links, and J -graded colorings of links. In this subsection we provide some definitions necessary to define the gradings. We begin with a first definition:

Definition 2.3. If L is an embedded link in Y^3 , we say that a pair (π, J) is a **grading assignment** of L if J is a finite set, and $\pi : C(L) \rightarrow J$ is a map from the set of components of L to J . Analogously, if Σ is an embedded surface in W , we say that a **grading assignment** of Σ is a map $\pi : C(\Sigma) \rightarrow J$.

For notational convenience, if s is a point on L we will sometimes write $\pi(s)$ for $\pi(K)$ where K is the component of L which contains s .

Definition 2.4. Suppose that L is a link in Y and $\pi : L \rightarrow J$ is a grading assignment. We say that L is **J -null-homologous** if

$$[\pi^{-1}(j)] = 0 \in H_1(Y; \mathbb{Z}),$$

for each $j \in J$.

Analogously, we could consider links L which are **rationally J -null-homologous**, i.e.

$$[\pi^{-1}(j)] = 0 \in H_1(Y; \mathbb{Q})$$

for each $j \in J$. For expository reasons we will focus on integrally J -null-homologous links.

Remark 2.5. It is important that our links be oriented. A link can be J -null-homologous with one orientation, but not with another. For example consider $L = S^1 \times \{p_1, p_2\} \subseteq S^1 \times S^2$. For the grading assignment sending both components to a single grading, L is J -null-homologous if the two components are given opposite orientations, but is not J -null-homologous if they are given the same orientation.

As we must color the link Floer complexes to get functorial cobordism maps, we also need to describe which colorings will allow for the colored modules $CFL^\infty(Y, \mathbb{L}, \sigma, \mathfrak{P}, \mathfrak{s})$ to admit Alexander and Maslov gradings.

Definition 2.6. If $(L, \mathbf{w}, \mathbf{z})$ is a multi-based link in Y , we say a tuple $(\sigma, \mathfrak{P}, \hat{\pi}, \pi, J)$ is a **J -graded coloring** if the following holds:

- (1) (σ, \mathfrak{P}) is a coloring of \mathbb{L} and (π, J) is a grading assignment of \mathbb{L} ;
- (2) the set of colors \mathfrak{P} is partitioned into two subsets $\mathfrak{P} = \mathfrak{P}_{\mathbf{w}} \sqcup \mathfrak{P}_{\mathbf{z}}$;
- (3) $\sigma(\mathbf{w}) \subseteq \mathfrak{P}_{\mathbf{w}}$ and $\sigma(\mathbf{z}) \subseteq \mathfrak{P}_{\mathbf{z}}$;
- (4) $\hat{\pi} : \mathfrak{P} \rightarrow J$ is a map such that

$$\pi \circ \iota = \hat{\pi} \circ \sigma,$$

where $\iota : \mathbf{w} \cup \mathbf{z} \rightarrow C(L)$ is the map induced by inclusion.

Example 2.7. One natural coloring that is graded is obtained by collapsing all of $U_{\mathbf{w}}$ to a single variable U , and collapsing all of the $V_{\mathbf{z}}$ variables to a single V . The set J is $\{*\}$ and $\hat{\pi}$ and π map every color and basepoint to $*$.

Another coloring of interest is the “tautological coloring”, obtained by setting $\mathfrak{P} = \mathbf{w} \cup \mathbf{z}$, and defining σ to be the identity map. The tautological coloring determines a J -graded coloring with $J = C(L)$, the set of components of L .

Similarly we can define J -graded colorings of surfaces with divides, for which the link cobordism maps of [Zem16b] will be graded and satisfy several grading change formulas:

Definition 2.8. Suppose that $F = (\Sigma, \mathcal{A})$ is a surface with divides, and let $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$ be the two subsurfaces of Σ , which meet along \mathcal{A} , with their designation as type \mathbf{w} or \mathbf{z} . A **grading assignment** of F is a map $\pi : C(\Sigma) \rightarrow J$. A J -graded coloring of F is a tuple $(\sigma, \mathfrak{P}, \hat{\pi}, \pi, J)$ such that

- (1) (σ, \mathfrak{P}) is a coloring of the components of $\Sigma \setminus \mathcal{A}$ and (π, J) is a grading assignment of F ;
- (2) the set of colors \mathfrak{P} is partitioned into two subsets $\mathfrak{P} = \mathfrak{P}_{\mathbf{w}} \sqcup \mathfrak{P}_{\mathbf{z}}$;
- (3) $\sigma(C(\Sigma_{\mathbf{w}})) \subseteq \mathfrak{P}_{\mathbf{w}}$ and $\sigma(C(\Sigma_{\mathbf{z}})) \subseteq \mathfrak{P}_{\mathbf{z}}$;
- (4) $\hat{\pi} : \mathfrak{P} \rightarrow J$ is a map such that

$$\pi \circ \iota = \hat{\pi} \circ \sigma,$$

where $\iota : C(\Sigma_{\mathbf{w}}) \cup C(\Sigma_{\mathbf{z}}) \rightarrow C(L)$ denotes inclusion.

We now define a notion of Seifert surface for J -null-homologous links.

Definition 2.9. A J -**Seifert surface** S for a J -null-homologous link L with grading assignment $\pi : L \rightarrow J$ is a collection of oriented, immersed surfaces $S_j \subseteq Y$, such that

$$\partial S_j = -\pi^{-1}(j),$$

where $\pi^{-1}(j) \subseteq L$ is given the orientation of L .

The various surfaces S_j of a J -Seifert surface will of course intersect, in general.

We remark that under our orientation conventions, a Seifert surface can be thought of as determining a link cobordism from $\pi^{-1}(j)$ to the empty link.

3. HEEGAARD TRIPLES AND Spin^c STRUCTURES

In this section we describe some constructions using Heegaard triples.

3.1. Constructing 4-manifolds and Spin^c structures from Heegaard triples. In [OS04b], Ozsváth and Szabó associate to a singly pointed Heegaard triple $(\Sigma, \alpha, \beta, \gamma, w)$, a 4-manifold with three boundary components, denoted $X_{\alpha\beta\gamma}$. The same construction works for Heegaard triples with multiple \mathbf{w} -basepoints. To a Heegaard triple that has two types of basepoints, \mathbf{w} and \mathbf{z} , we describe an embedded surface $\Sigma_{\alpha\beta\gamma}$ inside of the 4-manifold $X_{\alpha\beta\gamma}$.

Definition 3.1. We say that a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, \mathbf{w}, \mathbf{z})$ is a **doubly multi-pointed Heegaard triple** if each component of $\Sigma \setminus \alpha$ has exactly one \mathbf{w} -basepoint, and exactly one \mathbf{z} -basepoint, and the same holds for the β - and γ -curves.

Using the construction of Ozsváth and Szabó, we can construct a 4-manifold $X_{\alpha\beta\gamma}$ as the union

$$X_{\alpha\beta\gamma} = ((\Sigma \times \Delta) \cup (e_{\alpha} \times U_{\alpha}) \cup (e_{\beta} \times U_{\beta}) \cup (e_{\gamma} \times U_{\gamma})) / \sim,$$

where Δ is the triangle with edges (in clockwise order) e_{α}, e_{β} and e_{γ} . Also U_{α}, U_{β} and U_{γ} are genus $g(\Sigma)$ handlebodies with boundary Σ , which have α, β , and γ as the boundaries of compressing curves (respectively). The 4-manifold $X_{\alpha\beta\gamma}$ naturally has boundary

$$\partial X_{\alpha\beta\gamma} = -Y_{\alpha\beta} \sqcup -Y_{\beta\gamma} \sqcup Y_{\alpha\gamma}.$$

If $(\Sigma, \alpha, \beta, \gamma, \mathbf{w}, \mathbf{z})$ is a doubly multi-pointed Heegaard triple with two types of basepoints, then there is a naturally defined, properly embedded surface $\Sigma_{\alpha\beta\gamma}$ inside of $X_{\alpha\beta\gamma}$. It is formed as the union

$$\Sigma_{\alpha\beta\gamma} = ((-\mathbf{w} \cup \mathbf{z}) \times \Delta) \cup (e_{\alpha} \times f_{\alpha}) \cup (e_{\beta} \times f_{\beta}) \cup (e_{\gamma} \times f_{\gamma}).$$

Here $f_\alpha, f_\beta, f_\gamma$ are compact 1-manifolds, defined as follows. Inside of the handlebody U_α , one takes a Morse function which is 0 on Σ , has maximum value of 1, such that the intersection of Σ with the descending manifolds of the index 2 critical points of Σ are the α curves. One then takes f_α to be the union of the (finitely many) flowlines which intersect a basepoint in $\mathbf{w} \cup \mathbf{z} \subseteq \Sigma$. The 1-manifolds f_β and f_γ are defined similarly.

Note that the ends of $X_{\alpha\beta\gamma}$ are $-Y_{\alpha\beta}, -Y_{\beta\gamma}$ and $Y_{\alpha\gamma}$. Inside of each of these closed three manifolds is a link, defined by $\mathbb{L}_{\tau\sigma} = f_\tau \cup f_\sigma$. The link $\mathbb{L}_{\tau\sigma}$ is oriented in $Y_{\tau\sigma}$ by assuming that the intersections with Σ are negative at the \mathbf{w} -basepoints, and positive at the \mathbf{z} -basepoints. We include a picture in Figure 3.1.

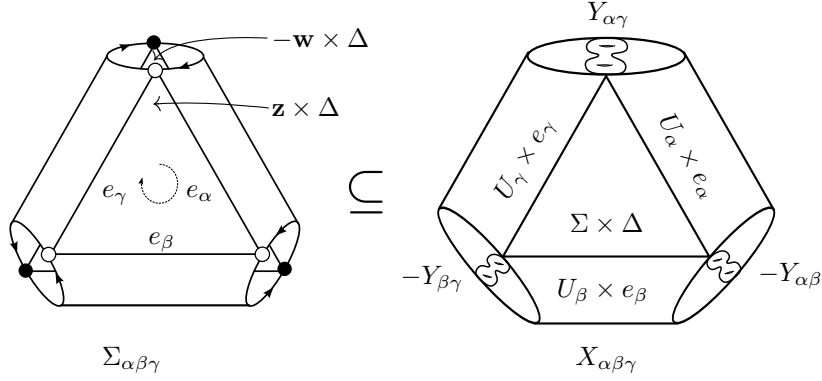


FIGURE 3.1. The surface $\Sigma_{\alpha\beta\gamma}$ inside of $X_{\alpha\beta\gamma}$. Orientations are shown. Note that the picture is somewhat simplistic, since there could be many basepoints, and the number of components on the ends need not be equal.

Using the orientation induced by an outward pointing normal vector N , (i.e. $v \in TL$ is positive if (v, N) is a positive basis of $T\Sigma_{\alpha\beta\gamma}$) we have that

$$\partial\Sigma_{\alpha\beta\gamma} = -L_{\alpha\beta} \sqcup -L_{\beta\gamma} \sqcup L_{\alpha\gamma}.$$

Now if $(\Sigma, \alpha, \beta, \gamma, \mathbf{w}, \mathbf{z})$ is a triple which is subordinate to β -bouquet of a framed 1-dimension link $\mathbb{S}_1 \subseteq Y_{\alpha\beta} \setminus L_{\alpha\beta}$, then the cobordism $W(Y_{\alpha\beta}, \mathbb{S}_1)$ can be obtained from $X_{\alpha\beta\gamma}$ by filling in $Y_{\beta\gamma}$ with 3- and 4-handles (see Section 3.2 for the definition of a β -bouquet). Inside of $W(Y_{\alpha\beta}, \mathbb{S}_1)$, there is a surface $L_{\alpha\beta} \times [0, 1]$, which is equal to $\Sigma_{\alpha\beta\gamma}$ after removing the intersection with the 3-handles and 4-handles (see [Juh16, Proposition 6.6])

We now make a remark on orientations. We note that our orientation conventions specify that $Y \times [0, 1]$, as a cobordism from $Y \times \{0\}$ to $Y \times \{1\}$, gets the product orientation. We make the same definition for surfaces. The reader should beware of the following though. To define an orientation on the 4-manifold $X_{\alpha\beta\gamma}$, the triangle Δ , viewed as a triangle with edges e_α, e_β , and e_γ (appearing in that order, counterclockwise), is given the clockwise orientation, as a manifold with boundary. This is somewhat unfortunate, as the triangle maps counts holomorphic maps into $\Sigma \times \Delta$, but Δ is given the counterclockwise orientation, in that context. Hopefully this isn't too confusing. It is important to make this distinction, since it affects formulas arising from intersection numbers.

Let $\mathcal{P}_{\alpha\beta\gamma}$ denote the set 2-chains on a triple $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ which have boundary equal to a linear combination of α, β and γ curves. We can define a map from $\mathcal{P}_{\alpha\beta\gamma}$ to $H_2(X_{\alpha\beta\gamma}; \mathbb{Z})$ as in [OS04b, Proposition 8.3] or [OS06, pg. 9]. This is defined by taking such a domain \mathcal{D} in Σ , and including it into $\{pt\} \times \Sigma \subseteq X_{\alpha\beta\gamma}$. We then extend the 2-chain outward towards the boundary of Δ , and then cap off with disks in U_α, U_β and U_γ to get a closed 2-chain $H(\mathcal{D}) \in H_2(X_{\alpha\beta\gamma}; \mathbb{Z})$. For Heegaard triples with a single basepoint, such domains which also don't pass over the basepoints are usually called **triple periodic domains**. In this paper we will call any domain in $\mathcal{P}_{\alpha\beta\gamma}$ a triple periodic domain if it has boundary equal to a sum of the α, β and γ curves. We note that it is easy to compute the intersection of $H(\mathcal{D})$ and $\Sigma_{\alpha\beta\gamma}$, since they intersect only at

$(\mathbf{w} \cup \mathbf{z}) \times \{pt\}$, and the multiplicity of the intersection points is given by the multiplicity of the domain \mathcal{D} at the basepoints. Noting the above remarks about orientations, we have that:

$$(2) \quad \langle PD[H(\mathcal{D})], [\Sigma_{\alpha\beta\gamma}] \rangle = \#(H(\mathcal{D}) \cap \Sigma_{\alpha\beta\gamma}) = n_{\mathbf{z}}(\mathcal{D}) - n_{\mathbf{w}}(\mathcal{D}),$$

and indeed, more generally, if Σ_j is a union of connected components of $\Sigma_{\alpha\beta\gamma}$, then the analogous equation holds, if we only sum over the basepoints in \mathbf{w} and \mathbf{z} corresponding to Σ_j .

In [OS04b], to a Heegaard diagram $(\Sigma, \alpha, \beta, \mathbf{w})$ for Y , Ozsváth and Szabó describe a map

$$\mathfrak{s}_{\mathbf{w}} : \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta} \rightarrow \text{Spin}^c(Y).$$

One starts with the upward gradient vector field for a Morse function inducing the Heegaard diagram. In a neighborhood of the flowlines passing through \mathbf{w} , and the intersection points of \mathbf{x} , one modifies the vector field so that it is nonvanishing (which can be done since these chosen flowlines connect critical points of opposite parities). This of course depends on the choice of basepoints \mathbf{w} , though we have the following:

Lemma 3.2. *If $\mathcal{H} = (\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ is a diagram for $(Y, (L, \mathbf{w}, \mathbf{z}))$, and $\mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$, then*

$$\mathfrak{s}_{\mathbf{w}}(\mathbf{x}) - \mathfrak{s}_{\mathbf{z}}(\mathbf{x}) = PD[L].$$

Proof. This follows from [OS04b, Lemma 2.19]. \square

Analogously, to a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, \mathbf{w})$, in [OS04b, Section 8] Ozsváth and Szabó also describe a map

$$\mathfrak{s}_{\mathbf{w}} : \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{z}) \rightarrow \text{Spin}^c(X_{\alpha\beta\gamma}).$$

Again this depends on the choice of basepoints \mathbf{w} . As an analog of the above lemma, we have the following:

Lemma 3.3. *If $(\Sigma, \alpha, \beta, \gamma, \mathbf{w}, \mathbf{z})$ is a doubly multi-pointed Heegaard triple, then*

$$\mathfrak{s}_{\mathbf{w}}(\psi) - \mathfrak{s}_{\mathbf{z}}(\psi) = PD[\Sigma_{\alpha\beta\gamma}].$$

Proof. The proof is analogous to [OS04b, Lemma 2.19]. See [OS04b, Section 8] for a precise description of the map $\mathfrak{s}_{\mathbf{w}} : \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{z}) \rightarrow \text{Spin}^c(X_{\alpha\beta\gamma})$. Ozsváth and Szabó associate an oriented two plane field to a triangle ψ on $X_{\alpha\beta\gamma}$, on the complement of a contractible set. In Figure 5 of [OS04b], Ozsváth and Szabó describe a singular codimension 1 foliation on $X_{\alpha\beta\gamma}$. Let $B \subseteq \text{int } \Delta$ be a ball containing the image of the singular leaf of the foliation under the projection map $\Sigma \times \Delta \rightarrow \Delta$. Upon inspection, the two 2-plane fields associated to $\mathfrak{s}_{\mathbf{w}}$ and $\mathfrak{s}_{\mathbf{z}}$ differ only on a neighborhood of $\Sigma_{\alpha\beta\gamma}$ (though they are not defined on all of a neighborhood of $\Sigma_{\alpha\beta\gamma}$). Instead, they are defined on a neighborhood of a surface

$$\Sigma_{\alpha\beta\gamma}^0 = \Sigma_{\alpha\beta\gamma} \setminus (\Sigma \times B).$$

Define also

$$X_{\alpha\beta\gamma}^0 = X_{\alpha\beta\gamma} \setminus (\Sigma \times B).$$

By construction, on each leaf of the foliation on $X_{\alpha\beta\gamma} \setminus (\Sigma \times B)$ shown in Figure 5 of [OS04b], the 2-plane field of $\mathfrak{s}_{\mathbf{w}}(\psi)$ is the orthogonal complement of $T\Sigma_{\alpha\beta\gamma}^0$ inside of the leaf of the foliation. In fact, we can identify each leaf of the foliation away from $\Sigma \times B$ either as one of the closed manifolds $Y_{\sigma\tau}$ or one of the manifolds with boundary U_{τ} , for various τ and σ . On the other hand, the 2-plane field of $\mathfrak{s}_{\mathbf{z}}(\psi)$ is given by the orthogonal complement of Reeb surgery along the intersection of $\Sigma_{\alpha\beta\gamma}$ with the leaves of the foliation on $X_{\alpha\beta\gamma}$. This is shown in Figure 3.2. Let $\xi_{\mathbf{w}}$ and $\xi_{\mathbf{z}}$ denote the oriented 2-plane fields for $\mathfrak{s}_{\mathbf{w}}(\psi)$ and $\mathfrak{s}_{\mathbf{z}}(\psi)$ used to define the Spin^c structures. The vector bundles $\xi_{\mathbf{w}}^{\perp}$ and $\xi_{\mathbf{z}}^{\perp}$, being oriented 2-plane fields, naturally obtain the structure of a complex line bundle, and thus each of the pairs $(\xi_{\mathbf{w}}, \xi_{\mathbf{w}}^{\perp})$ and $(\xi_{\mathbf{z}}, \xi_{\mathbf{z}}^{\perp})$ give $TX_{\alpha\beta\gamma}|_{X_{\alpha\beta\gamma} \setminus C}$ an almost complex structure (for a contractible subset $C \subseteq X_{\alpha\beta\gamma}$), which uniquely determines a Spin^c structure extending over all of $X_{\alpha\beta\gamma}$ by an obstruction theory argument. The first Chern class of $\mathfrak{s}_{\mathbf{w}}(\psi)$ or $\mathfrak{s}_{\mathbf{z}}(\psi)$ evaluated on submanifolds $F \subseteq X_{\alpha\beta\gamma}$, which the 2-plane field is defined on, can be computed as the first Chern class of complex bundle $TX_{\alpha\beta\gamma}|_F$, evaluated on the fundamental class of F .

Since the 2-plane fields used to define $\mathfrak{s}_{\mathbf{w}}(\psi)$ and $\mathfrak{s}_{\mathbf{z}}(\psi)$ agree outside of a neighborhood of $\Sigma_{\alpha\beta\gamma}$, we know that $\mathfrak{s}_{\mathbf{w}}(\psi) - \mathfrak{s}_{\mathbf{z}}(\psi)$ is an element of $H^2(X_{\alpha\beta\gamma}; \mathbb{Z})$ which is supported in a neighborhood of $\Sigma_{\alpha\beta\gamma}$.

As described in [Lip06, Lemma 10.5], the restriction map $\text{Spin}^c(X_{\alpha\beta\gamma}) \rightarrow \text{Spin}^c(X_{\alpha\beta\gamma}^0)$ is injective, and since restriction of Spin^c structures is natural with respect to the action of H^2 , it is sufficient to show that $(\mathfrak{s}_{\mathbf{w}}(\psi) - \mathfrak{s}_{\mathbf{z}}(\psi))|_{N(\Sigma_{\alpha\beta\gamma}^0)}$ differ by the class $PD[\Sigma_{\alpha\beta\gamma}^0] \in H^2(N(\Sigma_{\alpha\beta\gamma}^0), \partial N(\Sigma_{\alpha\beta\gamma}^0); \mathbb{Z})$, where $N(\Sigma_{\alpha\beta\gamma}^0)$ is closed

unit disk bundle of the normal bundle, and $\partial N(\Sigma_{\alpha\beta\gamma}^0)$ is the unit sphere bundle. The Thom isomorphism theorem yields an isomorphism $H^2(N(\Sigma_{\alpha\beta\gamma}^0), \partial N(\Sigma_{\alpha\beta\gamma}^0); \mathbb{Z}) \cong H^0(\Sigma_{\alpha\beta\gamma}^0; \mathbb{Z})$. Noting that $H^*(\Sigma_{\alpha\beta\gamma}^0; \mathbb{Z})$ is torsion free, if $[F]$ is any element of $H_2(N(\Sigma_{\alpha\beta\gamma}^0), \partial N(\Sigma_{\alpha\beta\gamma}^0); \mathbb{Z})$, represented by an embedded surface F , transverse to $\Sigma_{\alpha\beta\gamma}^0$, it is sufficient to show that

$$(\mathfrak{s}_{\mathbf{w}}(\psi) - \mathfrak{s}_{\mathbf{z}}(\psi))|_{N(\Sigma_{\alpha\beta\gamma}^0)}[F] = \#(\Sigma_{\alpha\beta\gamma}^0 \cap F).$$

Noting as well that $c_1(\mathfrak{s}_{\mathbf{o}}(\psi) + H) = c_1(\mathfrak{s}_{\mathbf{o}}(\psi)) + 2H$ for $\mathbf{o} \in \{\mathbf{w}, \mathbf{z}\}$, it is further sufficient to show that

$$(c_1(\mathfrak{s}_{\mathbf{w}}(\psi)) - c_1(\mathfrak{s}_{\mathbf{z}}(\psi)))|_{(N(\Sigma_{\alpha\beta\gamma}^0), \partial N(\Sigma_{\alpha\beta\gamma}^0))}[F] = 2\#(\Sigma_{\alpha\beta\gamma}^0 \cap F).$$

By what we've said, this will imply that $\mathfrak{s}_{\mathbf{w}}(\psi) - \mathfrak{s}_{\mathbf{z}}(\psi) = PD[\Sigma_{\alpha\beta\gamma}]$. This can now be performed as a model computation.

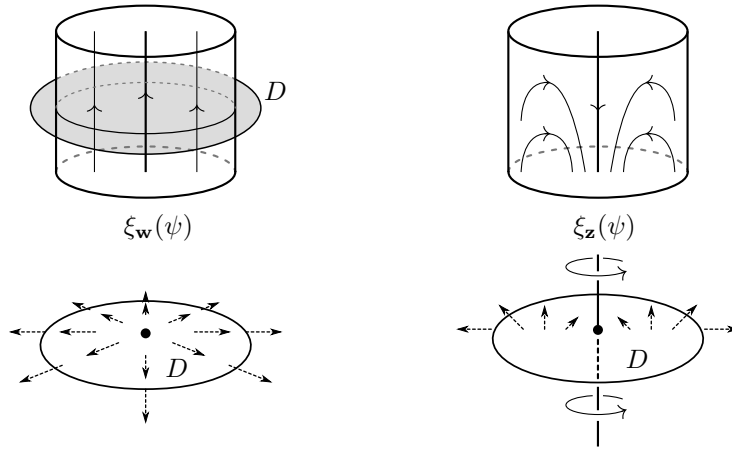


FIGURE 3.2. The 2-plane fields $\xi_{\mathbf{w}}(\psi)$ and $\xi_{\mathbf{z}}(\psi)$ are the orthogonal complements of the vector fields shown on the top row, which are shown inside of a (3-dimensional) leaf of the foliation. The disk $D \subseteq F$ is a neighborhood of the intersection point in $F \cap \Sigma_{\alpha\beta\gamma}^0$. On the bottom row, we show a sections of $\xi_{\mathbf{w}}(\psi)$ and $\xi_{\mathbf{z}}(\psi)$ on D . The section for $\xi_{\mathbf{w}}(\psi)$ is shown only on a diameter of D . To obtain it on all of D , one just rotates about the axis shown.

Suppose that F intersects $\Sigma_{\alpha\beta\gamma}^0$ transversely, and near any intersection points is tangent (positively or negatively) to the 2-plane field $\xi_{\mathbf{w}}$. Let $D \subseteq F$ be a disk centered at one of these intersection points. We can pick sections $v_{\mathbf{w}}$ and $v_{\mathbf{z}}$ of the 2-plane bundles $\xi_{\mathbf{w}}(\psi)$ and $\xi_{\mathbf{z}}(\psi)$ (respectively) which agree on ∂D , as follows. For $\xi_{\mathbf{w}}(\psi)$ we pick simply the outward radial vector field on D . For $\xi_{\mathbf{z}}(\psi)$, we take the section shown on a radius of D in Figure 3.2, and rotate around D to form a section $v_{\mathbf{z}}$. To get a well define section at the origin, one must scale $v_{\mathbf{z}}$ near the origin so that it vanishes there. The two sections $v_{\mathbf{w}}(\psi)$ and $v_{\mathbf{z}}(\psi)$ agree on ∂D , but the image of $v_{\mathbf{w}}$ in the total space of $\xi_{\mathbf{w}}(\psi)$ intersects the zero section transversely once, with sign $+1$ if the tangent space of D agrees with $\xi_{\mathbf{w}}(\psi)$ and with sign -1 if the tangent space of D is opposite the orientation of $\xi_{\mathbf{w}}(\psi)$. On the other hand, $v_{\mathbf{z}}$ intersects the zero section transversely once, but with opposite sign. Also note that one can (carefully) check that D has the same orientation as the 2-plane field for $\xi_{\mathbf{w}}(\psi)$ iff the corresponding point $F \cap \Sigma_{\alpha\beta\gamma}^0$ is a positive intersection point).

Finally, we note that $\xi_{\mathbf{w}}(\psi)^\perp$ and $\xi_{\mathbf{z}}(\psi)^\perp$ both admit the same non-vanishing section on a neighborhood of $\Sigma_{\alpha\beta\gamma}^0$ (a vector in the e_α, e_β , or e_γ direction), and hence the contributions to the Chern classes from these 2-plane bundles are equal.

We conclude that $(c_1(\mathfrak{s}_{\mathbf{w}}(\psi)) - c_1(\mathfrak{s}_{\mathbf{z}}(\psi)))[F] = 2\#(F \cap \Sigma_{\alpha\beta\gamma}^0)$ for any $F \in H_2(N(\Sigma_{\alpha\beta\gamma}^0), \partial N(\Sigma_{\alpha\beta\gamma}^0); \mathbb{Z})$, and we view $(c_1(\mathfrak{s}_{\mathbf{w}}(\psi)) - c_1(\mathfrak{s}_{\mathbf{z}}(\psi)))$ as a class in $H^2(N(\Sigma_{\alpha\beta\gamma}^0), \partial N(\Sigma_{\alpha\beta\gamma}^0); \mathbb{Z})$. We conclude

$$(\mathfrak{s}_{\mathbf{w}}(\psi) - \mathfrak{s}_{\mathbf{z}}(\psi))|_{X_{\alpha\beta\gamma}^0} = PD[\Sigma_{\alpha\beta\gamma}^0].$$

Hence

$$\mathfrak{s}_{\mathbf{w}}(\psi) - \mathfrak{s}_{\mathbf{z}}(\psi) = PD[\Sigma_{\alpha\beta\gamma}],$$

as elements of $\text{Spin}^c(X_{\alpha\beta\gamma})$. \square

3.2. Heegaard triples and bouquets of framed 1-dimensional links. Suppose that $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ is a multi-based link in Y . Let \mathbb{L}_α denote the arcs of $L \setminus (\mathbf{w} \cup \mathbf{z})$ which go from \mathbf{w} -basepoints to \mathbf{z} -basepoints. Let \mathbb{L}_β denote the arcs of $L \setminus (\mathbf{w} \cup \mathbf{z})$ which go from \mathbf{z} -basepoints to \mathbf{w} -basepoints.

Definition 3.4. A β -bouquet \mathcal{B}^β for a framed link \mathbb{S}_1 in $Y \setminus L$ is a collection of arcs which connect links in \mathbb{S}_1 to the interior of \mathbb{L}_β . We assume that there is exactly one arc per component of \mathbb{S}_1 , and each arc has one endpoint on \mathbb{S}_1 and one endpoint in \mathbb{L}_β .

An α -bouquet can be defined analogously. We will use β -bouquets to define the grading, but we will show in Section 7.3 that either α -bouquets or β -bouquets can be used.

Given a β -bouquet \mathcal{B}_β for a framed link $\mathbb{S}_1 \subseteq Y \setminus L$, we can consider the sutured manifold $Y(\mathcal{B}_\beta)$ obtained by removing regular neighborhoods of $L \cup \mathcal{B}_\alpha$ and $L \cup \mathcal{B}_\beta$, and adding sutures corresponding to the basepoints \mathbf{w} and \mathbf{z} . We have the following definition:

Definition 3.5. We say that a triple $(\Sigma, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n, \mathbf{w}, \mathbf{z})$ is **subordinate** to the β -bouquet \mathcal{B}_β for a framed 1-dimensional link $\mathbb{S}_1 \subseteq Y \setminus L$ if

- (1) $(\Sigma, \alpha_1, \dots, \alpha_n, \beta_{k+1}, \dots, \beta_n, \mathbf{w}, \mathbf{z})$ yields a sutured diagram for $Y(\mathcal{B}_\beta)$;
- (2) The curves β_1, \dots, β_k are each meridians of a different component of \mathbb{S}_1 and hence $(\Sigma, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \mathbf{w}, \mathbf{z})$ is a diagram for (Y, \mathbb{L}) ;
- (3) γ_{k+1} through γ_n are each small Hamiltonian isotopies of the curves $\beta_{k+1}, \dots, \beta_n$ respectively, with $|\beta_i \cap \gamma_j| = 2\delta_{ij}$;
- (4) for $i = 1, \dots, k$, the curve γ_i is induced by the framing of the corresponding link component that β_i is a meridian of.

For moving between Heegaard triples subordinate to a fixed bouquet, we have the following:

Lemma 3.6. Any two β -bouquets for \mathbb{S}_1 can be connected by the following moves:

- (1) isotopies and handleslides amongst the α -curves;
- (2) isotopies and handleslides amongst the curves $\{\beta_{k+1}, \dots, \beta_n\}$, while performing the analogous isotopy or handleslide of the curve $\{\gamma_{k+1}, \dots, \gamma_n\}$;
- (3) (1, 2)-stabilizations or destabilizations;
- (4) for $1 \leq i \leq k$, an isotopy of β_i , or a handleslide of β_i across a β_j with $k+1 \leq j \leq n$;
- (5) for $1 \leq i \leq k$, an isotopy of γ_i , or a handleslide of γ_i across a γ_j with $k+1 \leq j \leq n$;
- (6) A self diffeomorphism of $(Y, L \cup \mathcal{B}^\beta, \mathbf{w} \cup \mathbf{z})$ isotopic to the identity, through such diffeomorphisms.

Proof. This follows immediately from [Juh16, Lemma 6.5]. \square

Remark 3.7. Note that in general the surface $\Sigma_{\alpha\beta\gamma}$ associated to a Heegaard triple $\mathcal{T} = (\Sigma, \alpha, \beta, \gamma, \mathbf{w}, \mathbf{z})$ may be complicated and have high genus, though if \mathcal{T} is subordinate to an α - or β -bouquet for a framed 1-dimensional link, then the surface $\Sigma_{\alpha\beta\gamma}$ is diffeomorphic to a collection of triply punctured spheres.

4. KIRBY CALCULUS FOR MANIFOLDS WITH BOUNDARY

Our strategy for constructing an absolute grading on CFL° will parallel the construction of the absolute \mathbb{Q} -gradings on the groups $HF^\circ(Y, \mathfrak{s})$ in [OS06]. We will define a notion of a Kirby diagram for 3-manifolds with an embedded, multi-based link (Y, \mathbb{L}) , essentially by presenting the link complements $Y \setminus N(\mathbb{L})$ as surgery on the standard unlink complement $S^3 \setminus N(\mathbb{U})$ for an unlink \mathbb{U} in S^3 , then consider a Kirby calculus argument for how two such presentations differ.

It will be useful for our purposes to first define a more general notion of surgery presentations, not specific to link complements:

Definition 4.1. If M and M' are oriented 3-manifolds with boundary and $\phi : \partial M \rightarrow \partial M'$ is a fixed, orientation preserving diffeomorphism, we say that **parametrized surgery data** for (M, M', ϕ) is a pair (\mathbb{S}_1, f) where $\mathbb{S}_1 \subseteq \text{int } M$ is a framed link and

$$f : M(\mathbb{S}_1) \rightarrow M'$$

is a diffeomorphism which extends ϕ .

It is not hard to see that if M and M' are connected, oriented 3-manifolds and $\phi : \partial M \rightarrow \partial M'$ is an orientation preserving diffeomorphism, then there exists parametrized surgery data for (M, M', ϕ) . This can be seen by the following argument (cf. [Rob97]). Using the diffeomorphism ϕ , we form the closed, oriented three manifold $-M \cup (\partial M \times [0, 1]) \cup M'$. This bounds a compact oriented 4-dimensional manifold W . We can view such a manifold as a cobordism of manifolds with boundary from M to M' , (a “special” cobordism in the language of [Juh16]). We think of M and M' as the “horizontal” parts of ∂W and $[0, 1] \times \partial M$ as the “vertical” part of the boundary. We can find a Morse function which is 0 on M , t on $\partial M \times [0, 1]$ and 1 on M' , which has only index 1, 2, and 3 critical points. Using a standard trick we can replace index 1 and 3 critical points with index 2 critical points (by changing the 4-manifold), to get a cobordism from M to M' which has a Morse function with only index 2 critical points. If we take a gradient like vector field on W which is $\partial/\partial t$ on $\partial M \times [0, 1]$, then the descending manifolds from the index 2 critical points yield a framed link \mathbb{S}_1 in M , and the Morse function and gradient like vector field determine a diffeomorphism $f : M(\mathbb{S}_1) \rightarrow M'$ which is well defined up to isotopy and extends ϕ .

In [Kir78], Kirby creates a set of moves between any two parametrized surgery decompositions of the form (S^3, Y, \emptyset) for a closed, oriented 3-manifold Y . The moves between any two decompositions were blow-ups, blow-downs, isotopies of f or \mathbb{S}_1 , and handleslides. In [FR79], Fenn and Rourke extended the calculus to arbitrary oriented 3-manifolds, instead of just S^3 , though an extra move is required which is supported in a solid torus. In [Rob97], this is extended to oriented 3-manifolds with boundary, as long as a diffeomorphism of the boundaries is fixed.

We note that in our definition of parametrized surgery data, there is a choice of diffeomorphism $f : M(\mathbb{S}_1) \rightarrow M'$. The diffeomorphism is important for our purposes. We note that a Kirby move between two framed links \mathbb{S}_1 and \mathbb{S}'_1 in M canonically yields a diffeomorphism $M(\mathbb{S}_1) \rightarrow M(\mathbb{S}'_1)$ as we describe in the following paragraph. In particular, if (\mathbb{S}_1, f) is parametrized surgery data, and \mathbb{S}'_1 is the result of one of the above moves on \mathbb{S}_1 , then there is a diffeomorphism $f' : M(\mathbb{S}'_1) \rightarrow M'$ which is canonically specified. In such a way, blow-ups, blow-downs and handleslides determine moves not only between framed links, but instead between collections of parametrized surgery data.

We now describe the canonical diffeomorphism from $M(\mathbb{S}_1)$ to $M(\mathbb{S}'_1)$ resulting from a Kirby move. A good account can be found in [GS99, pg. 160]. Suppose for the sake of demonstration that we are in the case of standard Kirby calculus, so $M = S^3$ and $\partial M = \partial M' = \emptyset$. Suppose that \mathbb{S}_1 and \mathbb{S}'_1 are two framed links, such that $\mathbb{S}'_1 = \mathbb{S}_1 \cup \{U\}$, where U is a ± 1 framed unknot which is contained in a ball $B \subseteq S^3 \setminus \mathbb{S}_1$. In this case, the manifolds $S^3(\mathbb{S}_1) \setminus B = S^3(\mathbb{S}'_1) \setminus B(U)$ are canonically diffeomorphic, via the identity map. Noting that B and $B(U)$ are both 3-balls with an identification of their boundaries (via the identity map), our diffeomorphism of $S^3(\mathbb{S}_1) \setminus B$ and $S^3(\mathbb{S}'_1) \setminus B(U)$ can be extended over B , uniquely up to isotopy, since any diffeomorphism of the 2-sphere extends over the 3-ball, uniquely up to isotopy. Hence we get a canonical diffeomorphism from $S^3(\mathbb{S}_1)$ to $S^3(\mathbb{S}'_1)$. Similarly if \mathbb{S}'_1 is the result of handlesliding a link component in \mathbb{S}_1 across another link component, then a diffeomorphism from $S^3(\mathbb{S}'_1)$ to $S^3(\mathbb{S}_1)$ is canonically specified as follows. Note that $S^3(\mathbb{S}'_1) \setminus H = S^3(\mathbb{S}_1) \setminus H'$, where H and H' are genus 2 handlebodies containing the link components involved in the handleslide. Now ∂H and $\partial H'$ are canonically identified, and H and H' are diffeomorphic by a diffeomorphism which extends this identification. But since H and H' are handlebodies, a diffeomorphism from H to H' is canonically specified up to isotopy by its restriction to the boundary since $\text{MCG}(H_g, \partial H_g) = \{*\}$, for H_g a 3-dimensional, genus g handlebody. In such a manner, a sequence of Kirby moves between two framed links \mathbb{S}_1 and \mathbb{S}'_1 canonically determines a diffeomorphism $M(\mathbb{S}_1)$ and $M(\mathbb{S}'_1)$. As a specific example, a diffeomorphism $\psi : M \rightarrow M$ which is the identity on ∂M may be presented as a sequence of Kirby moves on framed links in M , starting and ending at the empty link in $\text{int } M$.

We state the following version of the result from [Rob97]:

Theorem 4.2. *Any two pairs of parametrized surgery data for (M, M', ϕ) can be connected by the following moves:*

- (1) (Move \mathcal{O}_0): isotopies of f or \mathbb{S}_1 which fix ∂M ;
- (2) (Move \mathcal{O}_1): handleslides of link components amongst each other;
- (3) (Move \mathcal{O}_2): blow ups or blow downs along a ± 1 framed unknot in $\text{int } M$;
- (4) (Move \mathcal{O}_3): adding or removing two link components, one with zero framing, and one with arbitrary framing, inside of a solid torus, as shown in Figure 4.1.

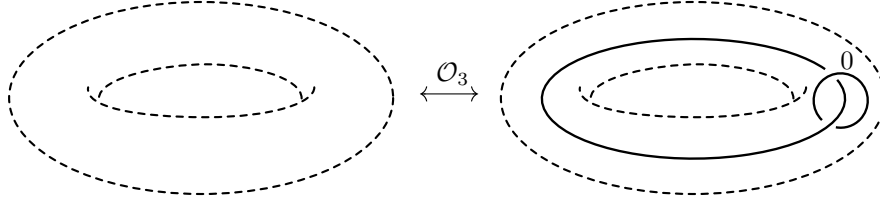


FIGURE 4.1. The move \mathcal{O}_3 , which takes place in a solid torus in M . The framing on the link component which is homotopically nontrivial in the solid torus can be arbitrary.

We note that in [Rob97], the result is stated without referencing the diffeomorphism f , though for our purposes, it is important to keep track of the diffeomorphism f . We now consider the implications of the previous theorem when applied to link complements.

Suppose that $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ is a multi-based link in a 3-manifold Y and let $\mathbb{U} = (U, \mathbf{w}_0, \mathbf{z}_0)$ denote a multi-based unlink in S^3 . Define

$$Y_L = Y \setminus N(L), \quad \text{and} \quad S_U^3 = S^3 \setminus N(U).$$

Let $\phi_0 : \mathbb{U} \rightarrow \mathbb{L}$ be a fixed homeomorphism of based links. We wish to describe diffeomorphisms from S_U^3 to Y_L . We add meridional sutures (closed 1-manifolds) $\gamma_{\mathbf{w}}$, $\gamma_{\mathbf{z}}$, $\gamma_{\mathbf{w}_0}$, and $\gamma_{\mathbf{z}_0}$ in the boundaries of Y_L and S_U^3 , corresponding to the basepoints of \mathbf{w} , \mathbf{z} , \mathbf{w}_0 and \mathbf{z}_0 .

A choice of framing λ of L specifies a diffeomorphism

$$\phi_\lambda : (\partial(S_U^3), \gamma_{\mathbf{w}_0}, \gamma_{\mathbf{z}_0}) \rightarrow (\partial(Y_L), \gamma_{\mathbf{w}}, \gamma_{\mathbf{z}}).$$

The diffeomorphism ϕ_λ is determined by ϕ_0 and λ up to isotopy preserving the circles $\gamma_{\mathbf{z}}$ and $\gamma_{\mathbf{w}}$. We make the following definition:

Definition 4.3. We call a tuple $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ a **parametrized Kirby diagram** for a multi-based link $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ in Y , if $\phi_0 : \mathbb{U} \rightarrow \mathbb{L}$ is a homeomorphism of multi-based 1-manifolds, from an unlink $\mathbb{U} = (U, \mathbf{w}, \mathbf{z})$ in S^3 to \mathbb{L} , λ is a framing of L , \mathbb{S}_1 is a framed link in S_U^3 , and

$$f : (S_U^3)(\mathbb{S}_1) \rightarrow Y_L$$

is a diffeomorphism which extends the diffeomorphism ϕ_λ described in the previous paragraph.

We will describe gradings

$$A_{Y, \mathbb{L}, S, \mathbb{P}}$$

where S is a choice of J -Seifert surface, \mathbb{P} is a parametrized Kirby decomposition for (Y, \mathbb{L}) . We will show that the above grading depends only on (Y, \mathbb{L}) and S .

We now consider a new move, which we will call \mathcal{O}'_3 . Given a framed link (L, λ) in Y , consider the effect of performing ± 1 surgery on a knot which encircles a component of L , as in Figure 4.2. Suppose $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ is a choice of parametrized Kirby diagram for (Y, \mathbb{L}) , with $f : S_U^3(\mathbb{S}_1) \rightarrow Y_L$ a diffeomorphism extending ϕ_λ . We get a diffeomorphism

$$f_K : S_U^3(\mathbb{S}_1 \cup \{K\}) \rightarrow Y_L(f(K)).$$

There is a canonical diffeomorphism of $Y_L(f(K))$ with Y_L which is the identity outside of a solid torus K whose boundary intersects ∂Y_L in an annulus. Hence we get a diffeomorphism

$$f_K : S_U^3(\mathbb{S}_1 \cup \{K\}) \rightarrow Y_L.$$

Of course on the boundary f_K no longer restricts to ϕ_λ , but instead $\phi_{\lambda'}$, where λ' is a new framing which differs by ± 1 on the component that K encircled.

We call this move \mathcal{O}'_3 . After performing move \mathcal{O}'_3 , we get a new parametrized Kirby diagram $\mathbb{P}_K = (\phi_0, \lambda', \mathbb{S}_1 \cup K, f_K)$ for (Y, \mathbb{L}) .

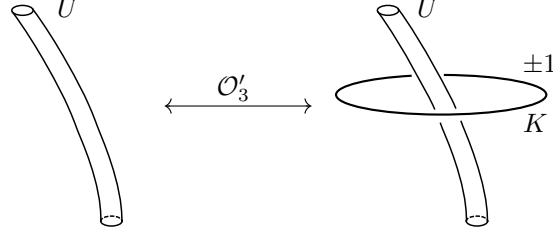


FIGURE 4.2. The move \mathcal{O}'_3 between two parametrized Kirby diagrams, and the configuration of the new component K of \mathbb{S}_1 with respect to the unlink $U \subseteq S^3$.

We now reformulate Theorem 4.2 to describe a sufficient set of moves between any two parametrized Kirby diagrams of a link:

Lemma 4.4. *Any two parametrized Kirby diagrams of a 3-manifold with a multi-based link can be connected by a sequence of the following moves:*

- (1) (Move \mathcal{O}_0): *Isotopies of f or \mathbb{S}_1 which preserve ∂Y_L , and also preserve $\gamma_{\mathbf{w}}$ and $\gamma_{\mathbf{z}}$;*
- (2) (Move \mathcal{O}_1): *handleslides amongst the components of \mathbb{S}_1 ;*
- (3) (Move \mathcal{O}_2): *blowing up or down along a ± 1 framed unknot which is null-isotopic in $S^3 \setminus (U \cup \mathbb{S}_1)$;*
- (4) (Move \mathcal{O}'_3): *blowing up along a ± 1 framed unknot which encircles a single component of U , exactly once, and no other components of U or \mathbb{S}_1 ;*
- (5) (Move \mathcal{O}_4): *Replacing $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ with $\mathbb{P}' = (\phi_0 \circ \psi, \lambda, \psi(\mathbb{S}_1), f \circ (\psi^{\mathbb{S}_1})^{-1})$, for $\psi : (S^3, \mathbb{U}) \rightarrow (S^3, \mathbb{U})$ which is orientation preserving for both S^3 and \mathbb{U} .*

Proof. For a fixed homeomorphism ϕ_0 and framing λ , Theorem 4.2 implies that Moves \mathcal{O}_0 , \mathcal{O}_1 , \mathcal{O}_2 and \mathcal{O}_3 suffice.

We first claim that for fixed ϕ_0 and λ , we can use only \mathcal{O}_3 moves where the solid torus encircles a single component of U , but is unlinked from all other components of \mathbb{S}_1 and U . Let us write \mathcal{O}'_3 this move. Note that we need to pay attention to both the framed links, as well as the parameterizing diffeomorphisms resulting from the move. Suppose \mathbb{S}_1 is a framed link in S^3_U and $f : S^3_U(\mathbb{S}_1) \rightarrow Y_L$ is a diffeomorphism. Let K denote a new link component (with any framing) and let us perform Move \mathcal{O}_3 on K , adding K (with some framing) and a zero framed unknot U_K encircling K to \mathbb{S}_1 . Let $\Phi_{\mathcal{O}_3, K}$ denote the diffeomorphism from $S^3_U(\mathbb{S}_1)$ to $S^3_U(\mathbb{S}_1 \cup \{K \cup U_K\})$ which is the identity outside of a solid torus containing K and U_K . Now let K' differ from K by a crossing change of K with itself, or a crossing change of K with another component of \mathbb{S}_1 . The link $\mathbb{S}_1 \cup \{K' \cup U_{K'}\}$ can be obtained by a handleslide of a link component across U_K (if the crossing change is of K with itself, then K is handleslid across U_K ; if the crossing change is of K with another component of \mathbb{S}_1 , then the other component is handleslide across U_K) followed by an isotopy of the framed links. Let $\Phi_H : S^3_U(\mathbb{S}_1 \cup \{K \cup U_K\}) \rightarrow S^3_U(\mathbb{S}_1 \cup \{K' \cup U_{K'}\})$ denote the diffeomorphism resulting from the composition of this handleslide and isotopy. We need to show that

$$(3) \quad \Phi_H \circ \Phi_{K, \mathcal{O}_3} \simeq \Phi_{K', \mathcal{O}_3}.$$

Suppose that we are handlesliding another component K_0 of \mathbb{S}_1 across U_K . Let $N(K_0)$ and $N(K)$ be regular neighborhoods of K_0 and K , inside of S^3_U , such that $N(K)$ contains U_K , as well. Let a be the arc over which we handleslide K_0 across U_K . Noting that $(N(K) \cup N(K_0) \cup N(a))(K_0, K, U_K)$ is a genus two handlebody, and that Φ_H and $\Phi_{K', \mathcal{O}_3} \circ \Phi_{K, \mathcal{O}_3}^{-1}$ both restrict to diffeomorphisms

$$(N(K) \cup N(K_0) \cup N(a))(K_0, K, U_K) \rightarrow (N(K) \cup N(K_0) \cup N(a))(K_0, K', U_{K'}),$$

which are the identity on the boundary, we conclude that they are isotopic, as $MCG(H_g, \partial H_g) = \{*\}$, where H_g is a genus g handlebody. Hence Equation (3) holds. This is shown in Figure 4.3. An analogous argument

holds for changing a crossing of K with itself. In that case, we let a be an arc from U_K to K , and consider a neighborhood N of $K \cup a$, which contains U_K . Now N is a genus two handlebody, and again, Φ_H and $\Phi_{K', \mathcal{O}_3} \circ \Phi_{K, \mathcal{O}_3}^{-1}$, differ only inside of N , and hence must be isotopic.

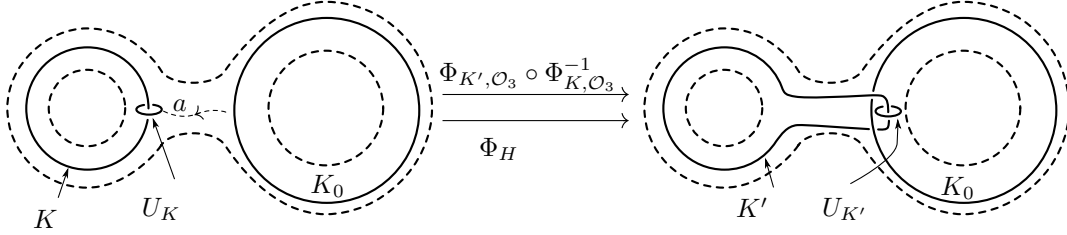


FIGURE 4.3. The diffeomorphisms Φ_H and $\Phi_{K', \mathcal{O}_3} \circ \Phi_{K, \mathcal{O}_3}^{-1}$ are equal outside of a genus two handlebody, and hence they are isotopic.

Hence in particular, instead of performing Move \mathcal{O}_3 on K , we can perform Move \mathcal{O}_3 on K' (the knot obtained by changing a crossing of K with either itself or with another component of \mathbb{S}_1), and then performing a handleslide, and the resulting framed link \mathbb{S}_1 and parameterizing diffeomorphism f will be the same.

Next, $K \subseteq S_U^3$ is a knot, and K' is the result of changing a crossing of K with U , then we need to show that the Move \mathcal{O}_3 performed on K can be written as a composition of Move \mathcal{O}_3 , performed on K' , as well as Moves $\mathcal{O}_0, \mathcal{O}_1$, and \mathcal{O}_2 and \mathcal{O}_3^0 . The procedure for doing this is shown in Figure 4.4. We perform Move \mathcal{O}_3^0 around U , then perform a sequence of handleslides, and then perform the inverse of Move \mathcal{O}_3^0 . The resulting framed links in S_U^3 that we perform surgery on are clearly equal. As before, the parameterizing diffeomorphism resulting from these two moves can be seen to be equal, since they agree outside of a genus two handlebody.

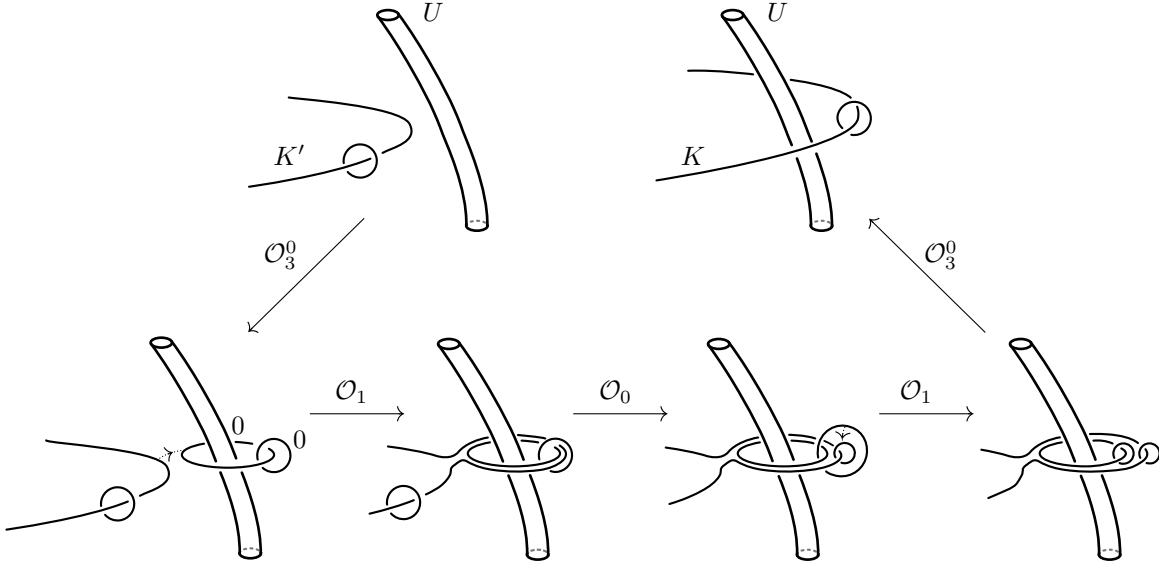


FIGURE 4.4. Move \mathcal{O}_3 performed on K , is equal to Move \mathcal{O}_3 performed on K' , composed with a sequence of Moves $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2$ and \mathcal{O}_3^0 .

In such manner, by reducing K to an unknot encircling just one (or no) components of U , as in Move \mathcal{O}_3^0 , we can write an arbitrary \mathcal{O}_3 move on a knot K , we can write an arbitrary \mathcal{O}_3 move as a composition of the moves $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2$ and \mathcal{O}_3^0 .

Finally we note that Move \mathcal{O}_3^0 can be written as a composition of two \mathcal{O}_3' moves (and possibly some of the other moves, depending on the framings).

Hence for a fixed ϕ_0 , Moves $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2$ and \mathcal{O}'_3 suffice. We finally note that any two ϕ_0 differ by precomposition with an orientation preserving diffeomorphism from \mathbb{U} to itself, and any such diffeomorphism extends to an orientation preserving diffeomorphism of (S^3, \mathbb{U}) with itself. Hence Move \mathcal{O}_4 is sufficient to move between any two ϕ_0 maps. \square

5. RELATIVE GRADINGS

In this section, we describe the three relative gradings, $A, \text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$, on $CFL^\infty(\mathcal{H}, \mathfrak{s})$, for a diagram \mathcal{H} of (Y, \mathbb{L}) .

5.1. The relative Maslov gradings. We now describe the two Maslov gradings, $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$. The grading $\text{gr}_{\mathbf{w}}$ is defined on generators by

$$(4) \quad \text{gr}_{\mathbf{w}}(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2 \sum_{w \in \mathbf{w}} n_w(\phi),$$

for a disk $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. The Chern class formula from [OS04a] shows that this is independent of the choice of disk ϕ , as long as $c_1(\mathfrak{s}_{\mathbf{w}}(\mathbf{x})) = c_1(\mathfrak{s})$ is torsion. We extend $\text{gr}_{\mathbf{w}}$ to the entire complex, by declaring all $U_{\mathbf{w}}$ variables to have grading change -2 , and all $V_{\mathbf{z}}$ variables to have grading change 0 .

Analogously, we can define a relative grading $\text{gr}_{\mathbf{z}}$. On generators it is defined as

$$(5) \quad \text{gr}_{\mathbf{z}}(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2 \sum_{z \in \mathbf{z}} n_z(\phi),$$

for a disk $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. We extend $\text{gr}_{\mathbf{z}}$ to the entire complex, by declaring all $V_{\mathbf{z}}$ variables to have grading change -2 , and all $U_{\mathbf{w}}$ variables to have grading change 0 . By the Chern class formula of Ozsváth and Szabó, this is independent of the disk in the case that $c_1(\mathfrak{s}_{\mathbf{z}}(\mathbf{x})) = c_1(\mathfrak{s} - PD[L])$ is torsion.

5.2. The relative Alexander multi-grading. Suppose that $\pi : L \rightarrow J$ is a grading assignment. We now define a relative \mathbb{Z}^J grading A on $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$, in the case that L is J -null-homologous. Given a disk ϕ , and an element $j \in J$, define

$$n_{\mathbf{z}}(\phi)_j = \sum_{\substack{z \in \mathbf{z} \\ \pi(z)=j}} n_z(\phi), \quad \text{and} \quad n_{\mathbf{w}}(\phi)_j = \sum_{\substack{w \in \mathbf{w} \\ \pi(w)=j}} n_w(\phi).$$

As usual, the grading is defined on generators as

$$(6) \quad A(\mathbf{x}, \mathbf{y})_j = (n_{\mathbf{z}} - n_{\mathbf{w}})(\phi)_j,$$

for a homology disk $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ and an index $j \in J$. This is extended to the entire complex $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$ by declaring $V_{\mathbf{z}}$ to be $+1$ graded in index $\pi(z)$, and declaring $U_{\mathbf{w}}$ to be -1 graded in index $\pi(w)$.

Of course the above formula depends *a-priori* on the choice of homology disk $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. We have the following (cf. eg. [OS04b, Lemma 2.18]):

Lemma 5.1. *If L is a link with a grading assignment $\pi : C(L) \rightarrow J$, then Equation (6) is independent of the disk ϕ iff \mathbb{L} is (rationally) J -null-homologous.*

Proof. Two disks in $\pi_2(\mathbf{x}, \mathbf{y})$ differ by splicing in elements $\pi_2(\mathbf{x}, \mathbf{x})$, so we just need to show that the expression $(n_{\mathbf{z}} - n_{\mathbf{w}})(\phi)_j$ vanishes for any element $\phi \in \pi_2(\mathbf{x}, \mathbf{x})$.

There is a surjective map

$$H : \pi_2(\mathbf{x}, \mathbf{x}) \rightarrow H_2(Y; \mathbb{Z})$$

given by sending the domain of a disk, a 2-chain on Σ with boundary equal to a linear combination of the α - and β -curves, to the 2-cycle in Y formed by taking $\mathcal{D}(\phi)$ and adding multiples of the compression disks in the α - and β - handlebodies, along the boundary curves of $\mathcal{D}(\phi)$. Given a class $\phi \in \pi_2(\mathbf{x}, \mathbf{x})$, the integer $(n_{\mathbf{z}} - n_{\mathbf{w}})(\phi)_j$ satisfies

$$(7) \quad (n_{\mathbf{z}} - n_{\mathbf{w}})(\phi)_j = \#(H(\phi) \cap \pi^{-1}(j)),$$

where $\#(H(\phi) \cap \pi^{-1}(j))$ is the oriented intersection number. The claim now follows. \square

Using the relative Alexander grading on the uncolored complexes, we can define an Alexander grading on the colored complexes, for J -graded colorings.

Lemma 5.2. *Suppose that \mathbb{L} is a multi-based link in Y and $(\sigma, \mathfrak{P}, \hat{\pi}, \pi, J)$ is a J -graded coloring of \mathbb{L} . If \mathbb{L} is J -null-homologous, then there is a well defined relative Alexander multi-grading on $CFL^\circ(Y, \mathbb{L}, \sigma, \mathfrak{P}, \mathfrak{s})$, taking values in \mathbb{Z}^J . Recalling that $\mathfrak{P} = \mathfrak{P}_{\mathbf{w}} \sqcup \mathfrak{P}_{\mathbf{z}}$ by definition of a J -graded coloring, if $t \in \mathfrak{P}_{\mathbf{z}}$ then V_t is $+1$ graded if $t \in \mathfrak{P}_{\mathbf{z}}$ then U_t is -1 graded. Similarly there are well defined relative Maslov gradings $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ on the colored complexes.*

Proof. By definition of a J -graded coloring, we have $\mathfrak{P} = \mathfrak{P}_{\mathbf{w}} \sqcup \mathfrak{P}_{\mathbf{z}}$ and $\sigma(\mathbf{w}) \subseteq \mathfrak{P}_{\mathbf{w}}$ and $\sigma(\mathbf{z}) \subseteq \mathfrak{P}_{\mathbf{z}}$. If \mathbf{x} and \mathbf{y} are two intersection points, the quantity $A(\mathbf{x}, \mathbf{y})_j$ is equal to the same formula as before. Since the colors in \mathfrak{P} are partitioned into $\mathfrak{P}_{\mathbf{w}}$ and $\mathfrak{P}_{\mathbf{z}}$, and since the grading assignment $\pi : C(L) \rightarrow J$ satisfies

$$\pi \circ \iota = \hat{\pi} \circ \sigma$$

for a map $\hat{\pi} : \mathfrak{P} \rightarrow J$ (where $\iota : \mathbf{w} \cup \mathbf{z} \rightarrow C(L)$ is the natural inclusion), the relative grading has a well defined extension to monomials with nonzero powers of the variables. Similar considerations apply for the Maslov gradings. \square

Lemma 5.3. *Given a J -null-homologous link, the subset of $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$ generated by monomials of a particular relative Alexander grading are subcomplexes.*

Proof. This is an easy computation. \square

Finally, we note that the relative Alexander grading is preserved by the change of diagrams maps:

Lemma 5.4. *Suppose \mathbb{L} is a link in Y , with a grading assignment $\pi : C(L) \rightarrow J$, and that \mathbb{L} is J -null-homologous. If (\mathcal{H}, J_s) and (\mathcal{H}', J'_s) are two choices of diagrams and almost complex structures for (Y, \mathbb{L}) , and $\mathfrak{s} \in \text{Spin}^c(Y)$ is a torsion Spin^c structure, then the change of diagrams map*

$$\Phi_{(\mathcal{H}, J_s) \rightarrow (\mathcal{H}', J'_s)} : CFL_{J_s}^\circ(\mathcal{H}, \mathfrak{s}) \rightarrow CFL_{J'_s}^\circ(\mathcal{H}', \mathfrak{s})$$

preserves the relative Alexander multi-grading over \mathbb{Z}^J , and the two relative Maslov gradings.

Proof. To prove this, we check the statement for the maps associated to isotopies and handleslides of the α - and β -curves, $(1, 2)$ -stabilizations and destabilizations, diffeomorphisms of the 3-manifold, and changes of the almost complex structure. Let us consider a handleslide or isotopy of the α -curves. We will leave the other moves to the reader. Suppose that $(\Sigma, \alpha', \alpha, \beta)$ is a triple for an α -equivalence, and suppose that \mathbf{x} and \mathbf{x}' are two intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Suppose that $\psi \in \pi_2(\theta, \mathbf{x}, \mathbf{y})$ and $\pi_2(\theta', \mathbf{x}', \mathbf{y}')$ are two triangles with Maslov index zero, with θ and θ' in the same Alexander and Maslov gradings as the top generator of $\mathcal{HFL}^\circ(\Sigma, \alpha', \alpha)$ (for $\circ \in \{\wedge, -\}$). Now since ψ and ψ' represent the same Spin^c structure, there are disks

$$\phi_{\alpha'\alpha} \in \pi_2(\theta', \theta), \quad \phi_{\alpha\beta} \in \pi_2(\mathbf{x}', \mathbf{x}), \quad \text{and} \quad \phi_{\alpha'\beta} \in \pi_2(\mathbf{y}, \mathbf{y}')$$

such that

$$\psi' = \psi + \phi_{\alpha'\alpha} + \phi_{\alpha\beta} + \phi_{\alpha'\beta}.$$

Hence

$$\begin{aligned} A(\mathbf{y} \cdot U_{\mathbf{w}}^{n_{\mathbf{w}}(\psi)} V_{\mathbf{z}}^{n_{\mathbf{z}}(\psi)}, \mathbf{y}' \cdot U_{\mathbf{w}}^{n_{\mathbf{w}}(\psi')} V_{\mathbf{z}}^{n_{\mathbf{z}}(\psi')})_j &= (n_{\mathbf{z}} - n_{\mathbf{w}})(\phi_{\alpha'\beta} + \psi - \psi')_j \\ &= (n_{\mathbf{z}} - n_{\mathbf{w}})(-\phi_{\alpha\beta})_j - (n_{\mathbf{z}} - n_{\mathbf{w}})(\phi_{\alpha'\alpha})_j \\ &= A(\mathbf{x}, \mathbf{x}')_j, \end{aligned}$$

since $(n_{\mathbf{z}} - n_{\mathbf{w}})(\phi_{\alpha'\alpha})_j = 0$ as θ and θ' are in the same Alexander grading. Hence $\Phi_{\beta}^{\alpha \rightarrow \alpha'}$ preserves the relative Alexander multi-grading. Easy adaptations of the above argument show that $\Phi_{\beta}^{\alpha \rightarrow \alpha'}$ preserves the other relative gradings, and also that the maps associated to β -equivalences, changes of almost complex structure, $(1, 2)$ -stabilization and diffeomorphisms of Y all preserve the relative gradings. \square

5.3. Two simple examples. We briefly consider a few examples, focusing on cases where one grading may be defined but not another.

Example 5.5. Consider $Y = S^1 \times S^2$, with $K = S^1 \times \{pt\}$. A diagram is shown in Figure 5.1. For \mathfrak{s} the torsion Spin^c structure, we see that

$$\mathcal{CFL}^-(Y, \mathbb{L}, \mathfrak{s}) \cong \mathbb{Z}_2[U, V] \xrightarrow{1+V} \mathbb{Z}_2[U, V].$$

The homology is $\mathbb{Z}[U, V]/(V - 1)$. The grading $\text{gr}_{\mathbf{w}}$ is defined, but $\text{gr}_{\mathbf{z}}$ cannot be defined (for example V acts by the identity on homology). Note that $\mathfrak{s} - PD[K]$ is not torsion, so we wouldn't expect $\text{gr}_{\mathbf{z}}$ to be well defined.

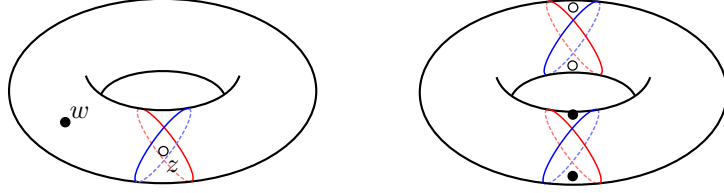


FIGURE 5.1. Two diagrams for $S^1 \times S^2$, with a knot or link. The left is for the knot $K = S^1 \times \{p\}$ and the right is for $K = S^1 \times \{p_1, p_2\}$, with the components going in opposite directions. All intersection points are mapped to the torsion Spin^c structure by $\mathfrak{s}_{\mathbf{w}}$.

Example 5.6. Another instructive example is $S^1 \times S^2$ with $L = S^1 \times \{p_1, p_2\}$, giving each component two basepoints, and the two components opposite orientations. A diagram is shown on the right side of Figure 5.1. The intersection points represent the torsion Spin^c structure with respect to $\mathfrak{s}_{\mathbf{w}}$. It's easy to see that the collapsed Alexander grading can be defined, but a two component Alexander grading cannot be defined. One can get a diagram for the link where both components are going in the same direction by switching the roles of the two \mathbf{w} and \mathbf{z} basepoints on a component. The intersection points then represent a Spin^c structure \mathfrak{s}_1 where $\langle c_1(\mathfrak{s}_1), \{pt\} \times S^2 \rangle = \pm 2$.

5.4. Absolute gradings on $\mathcal{CFL}^\circ((S^1 \times S^2)^{\#k}, \mathbb{U}, \mathfrak{s}_0)$ for unlinks \mathbb{U} . As a first step to describing the absolute gradings in general, we first declare the absolute gradings for unlinks in $(S^1 \times S^2)^{\#k}$, with various configurations of basepoints. It is sufficient to define the gradings on the generators of $\widehat{\mathcal{CFL}}$, since we can just extend over all elements of the full chain complex by declaring the grading of the variables. For simplicity assume that Σ is connected. For disconnected manifolds we define the grading of a generator as the sum of the gradings of the constituent generators on each component (note our convention is that $HF^-(S^3)$ has top degree generator in degree zero).

Lemma 5.7. *The \mathbb{Z}_2 -module $\widehat{HFL}((S^1 \times S^2)^{\#k}, \mathbb{U}, \mathfrak{s}_0)$ has rank $2^{|\mathbf{w}|+k-1}$. Furthermore, the group $\widehat{HFL}((S^1 \times S^2)^{\#k}, \mathbb{U}, \mathfrak{s}_0)$ has a top degree generator with respect to each of the gradings $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$, for which we write $\Theta^{\mathbf{w}}$ or $\Theta^{\mathbf{z}}$.*

Proof. By Lemma 5.4, the relatively graded isomorphism type of the group $\widehat{HFL}((S^1 \times S^2)^{\#k}, \mathbb{U}, \mathfrak{s}_0)$ is an invariant, so we need only check the claim for a particularly simple diagram. One can prove this by induction. A new unknot component with exactly two basepoints can be added by adding it in a sphere disjoint from $(S^1 \times S^2)^{\#k}$, and then attaching it with a 1-handle. It's easy to verify that the new complex has two top generators $\Theta^{\mathbf{w}}$ and $\Theta^{\mathbf{z}}$, since the old complex did. Using, for example, the holomorphic disk computation from [Zem16a, Proposition 5.3] of a quasi-stabilized differential, one can see that the same is true when we add two basepoints to a link component. \square

For an arbitrary diagram $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ of $((S^1 \times S^2)^{\#k}, \mathbb{U})$, we can thus define absolute Maslov and Alexander gradings by declaring the values of the gradings on $\Theta^{\mathbf{w}}$. Note that it is sufficient to define the Alexander \mathbb{Z}^J grading when $J = C(\mathbb{U})$ and the grading assignment $\pi : C(\mathbb{U}) \rightarrow J$ is the identity map. For

any other choice of π and J , we define the absolute grading over J to be obtained by collapsing and extending gradings, i.e. if $\pi : C(\mathbb{U}) \rightarrow J$, then we define

$$(8) \quad A_{(S^1 \times S^2)^{\#k}, \mathbb{U}, \pi}(\mathbf{x})_j = \sum_{K \in \pi^{-1}(j)} A_{(S^1 \times S^2)^{\#k}, \mathbb{U}}(\mathbf{x})_K.$$

We declare

$$(9) \quad \text{gr}_{\mathbf{w}}(\Theta^{\mathbf{w}}) = \text{gr}_{\mathbf{z}}(\Theta^{\mathbf{z}}) = \frac{1}{2}(k + |\mathbf{w}| - 1) = \frac{1}{2}(k + |\mathbf{z}| - 1),$$

where $|\mathbf{w}| = |\mathbf{z}|$ is the total number of \mathbf{w} -basepoints. Similarly we declare that

$$(10) \quad A_{(S^1 \times S^2)^{\#k}, \mathbb{U}, S}(\Theta^{\mathbf{w}})_K = \frac{1}{2}(n_K - 1),$$

where $K \in J = C(\mathbb{U})$ is a component of \mathbb{U} and n_K is half the total number of basepoints on K (i.e. the number of \mathbf{w} -basepoints on K). From now on, we will write only A_K for $(A_{(S^1 \times S^2)^{\#k}, \mathbb{U}, S})_K$. Note that these declarations determine $\text{gr}_{\mathbf{w}}, \text{gr}_{\mathbf{z}}$ and A_K on all of $CFL^\circ((S^1 \times S^2)^{\#k}, \mathbb{U}, \mathfrak{s}_0)$.

If \mathbb{U} has exactly two basepoints per component, then it's easy to see that $\Theta^{\mathbf{w}} = \Theta^{\mathbf{z}}$, and hence we have $\text{gr}_{\mathbf{w}} = \text{gr}_{\mathbf{z}}$ and $A_K = 0$ on $\widehat{HFL}((S^1 \times S^2)^{\#k}, \mathbb{U}, \mathfrak{s}_0)$.

Computing for diagrams which are obtained by performing a sequence of quasi-stabilizations, we can compute the following equations for a general configuration of basepoints on $\mathbb{U} \subseteq (S^1 \times S^2)^{\#k}$:

$$\begin{aligned} \text{gr}_{\mathbf{w}}(\Theta^{\mathbf{z}}) &= \text{gr}_{\mathbf{w}}(\Theta^{\mathbf{w}}) - \sum_{K \in C(\mathbb{U})} (n_K - 1) \\ \text{gr}_{\mathbf{z}}(\Theta^{\mathbf{w}}) &= \text{gr}_{\mathbf{z}}(\Theta^{\mathbf{z}}) - \sum_{K \in C(\mathbb{U})} (n_K - 1) \\ A(\Theta^{\mathbf{w}})_K &= \frac{1}{2}(n_K - 1) \\ A(\Theta^{\mathbf{z}})_K &= -\frac{1}{2}(n_K - 1). \end{aligned}$$

Finally we note that

$$(11) \quad A = \frac{1}{2}(\text{gr}_{\mathbf{w}} - \text{gr}_{\mathbf{z}}),$$

where A denotes the collapsed Alexander grading, since the equation is obviously true for the relative gradings, and is true for the absolute gradings evaluated at $\Theta^{\mathbf{w}}$.

6. COHERENT GRADINGS ON COHERENT CHAIN HOMOTOPY TYPES

The objects $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$ are not actual chain complexes. Instead they are “coherent chain homotopy types”, indexed by the set of admissible diagrams for (Y, \mathbb{L}) . Naturally we would like to make precise what it means to specify a grading on $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$, since the proof of invariance of the gradings does not factor through a single diagram. Additionally, as the change of diagrams are not isomorphisms, instead just chain homotopy equivalences on curved chain complexes, we need to be somewhat careful about pushing forward a grading along the change of diagrams maps.

For an admissible diagram \mathcal{H} of (Y, \mathbb{L}) , with grading assignment $\pi : C(L) \rightarrow J$, we define

$$\mathbb{A}(\mathcal{H}, \pi, \mathfrak{s}),$$

to be the set of absolute \mathbb{Q} -lifts of the relative Alexander grading defined in Section 5. Analogously, we define

$$\mathbb{G}_{\mathbf{w}}(\mathcal{H}, \mathfrak{s}), \quad \text{and} \quad \mathbb{G}_{\mathbf{z}}(\mathcal{H}, \mathfrak{s})$$

to be the set of absolute \mathbb{Q} -lifts of the relative Maslov gradings define in Section 5. In this section, we define a notion of a **coherent grading**, which is essentially one which respects the change of diagrams maps. For the set of coherent gradings for a grading assignment π , we will later introduce the notation $\vec{\mathbb{A}}(Y, \mathbb{L}, \pi, \mathfrak{s})$. We will define it as the direct limit over the sets $\mathbb{A}(\mathcal{H}, \pi, \mathfrak{s})$, using the change of diagrams maps which we define below. This is somewhat pedantic, but results in cleaner arguments later on.

6.1. Change of diagrams maps for Alexander gradings. Suppose \mathcal{H} and \mathcal{H}' are two admissible diagrams for (Y, \mathbb{L}) . Analogously to the change of diagrams maps for the chain complexes, we now define maps

$$\Phi_{\mathcal{H} \rightarrow \mathcal{H}'} : \mathbb{A}(\mathcal{H}, \pi, \mathfrak{s}) \rightarrow \mathbb{A}(\mathcal{H}', \pi, \mathfrak{s})$$

on gradings. These will be defined as a composition of maps associated to individual moves. If $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ is a diagram, and α' is another choice of α curves such that α' is the result of a sequence of handleslides or isotopies of the α curves and $(\Sigma, \alpha', \alpha, \beta)$ satisfies the appropriate admissibility requirements, then we can define a map

$$\Phi_{\beta}^{\alpha \rightarrow \alpha'} : \mathbb{A}(\Sigma, \alpha, \beta, \pi, \mathfrak{s}) \rightarrow \mathbb{A}(\Sigma, \alpha', \beta, \pi, \mathfrak{s}),$$

as follows. We note that $(\Sigma, \alpha', \alpha)$ is a diagram is for an unlink in $(S^1 \times S^2)^{\#k}$, with two basepoints on each component. If $A \in \mathbb{A}(\mathcal{H}, \pi, \mathfrak{s})$, we define $\Phi_{\beta}^{\alpha \rightarrow \alpha'}(A)$ by the formula

$$\Phi_{\beta}^{\alpha \rightarrow \alpha'}(A)(\mathbf{y})_j = A(\theta)_j + A(\mathbf{x})_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi)_j,$$

for a homology triangle $\psi \in \pi_2(\theta, \mathbf{x}, \mathbf{y})$ with $\mathfrak{s}_{\mathbf{w}}(\psi) = \mathfrak{s}|_{X_{\alpha'\alpha\beta}}$ under the identification of $X_{\alpha'\alpha\beta}$ as a subset of $Y \times [0, 1]$. The grading $\Phi_{\beta}^{\alpha \rightarrow \alpha'}(A)$ is independent of the choice of the intersection points \mathbf{x} and θ , as well as the homology triangle ψ , since any other homology triangle representing \mathfrak{s} with the same endpoints can be obtained by splicing in homology disks on $(\Sigma, \alpha', \alpha)$, (Σ, α, β) , and (Σ, α', β) which doesn't affect the formula.

Analogously, if β' differs from β by a sequence of handleslides or isotopies, and $(\Sigma, \alpha, \beta, \beta')$ achieves the appropriate admissibility hypothesis, we can define a map $\Phi_{\beta \rightarrow \beta'}^{\alpha}$.

We have the following:

Lemma 6.1. *Suppose that $(\Sigma, \alpha'', \alpha', \alpha, \beta)$ is an admissible quadruple and α, α' and α'' are all related to each other by a sequence of handleslides and isotopies. If $A \in \mathbb{A}(\Sigma, \alpha, \beta, \pi, \mathfrak{s})$, then*

$$\Phi_{\beta}^{\alpha \rightarrow \alpha''}(A) = (\Phi_{\beta}^{\alpha' \rightarrow \alpha''} \circ \Phi_{\beta}^{\alpha \rightarrow \alpha'})(A).$$

Proof. Suppose that $\mathbf{x} \in \mathbb{T}_{\alpha''} \cap \mathbb{T}_{\beta}$ and pick intersection points $\theta_{\alpha''\alpha'} \in \mathbb{T}_{\alpha''} \cap \mathbb{T}_{\alpha'}$, $\theta_{\alpha'\alpha} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\alpha}$, $\theta_{\alpha''\alpha} \in \mathbb{T}_{\alpha''} \cap \mathbb{T}_{\alpha}$, $\mathbf{x}_{\alpha\beta} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$ and $\mathbf{x}_{\alpha'\beta} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta}$, such that $\theta_{\alpha''\alpha}$, $\theta_{\alpha'\alpha}$ and $\theta_{\alpha''\alpha'}$ represent the torsion Spin^c structure. Pick triangles $\psi_{\alpha''\alpha'\alpha}$, $\psi_{\alpha''\alpha\beta}$, $\psi_{\alpha'\alpha\beta}$, and $\psi_{\alpha''\alpha'\beta}$ with the previous intersection points as vertices, such that

$$(12) \quad \psi_{\alpha''\alpha\beta} + \psi_{\alpha''\alpha'\alpha} = \psi_{\alpha'\alpha\beta} + \psi_{\alpha''\alpha'\beta}.$$

By definition, we have

$$\Phi_{\alpha \rightarrow \alpha''}^{\beta}(A)(\mathbf{x})_j = A(\mathbf{x}_{\alpha\beta})_j + A(\theta_{\alpha''\alpha})_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi_{\alpha''\alpha\beta})_j$$

and

$$(\Phi_{\beta}^{\alpha' \rightarrow \alpha''} \circ \Phi_{\beta}^{\alpha \rightarrow \alpha'})(A)(\mathbf{x})_j = A(\mathbf{x}_{\alpha\beta})_j + A(\theta_{\alpha'\alpha})_j + A(\theta_{\alpha''\alpha'})_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi_{\alpha''\alpha'\beta} + \psi_{\alpha'\alpha\beta})_j.$$

The second minus the first is

$$-A(\theta_{\alpha''\alpha})_j + A(\theta_{\alpha'\alpha})_j + A(\theta_{\alpha''\alpha'})_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi_{\alpha''\alpha'\alpha})_j,$$

which we need to show is zero. We note that the previous value is independent of the endpoints $\theta_{\alpha''\alpha}$, $\theta_{\alpha'\alpha}$ or $\theta_{\alpha''\alpha'}$, as well as the homology triangle $\psi_{\alpha''\alpha'\alpha}$, as any other homology triangle representing the same Spin^c structure on $X_{\alpha''\alpha'\alpha}$ can be obtained from $\psi_{\alpha''\alpha'\alpha}$ by splicing in homology disks on the three diagrams representing the ends of $X_{\alpha''\alpha'\alpha}$, and the above quantity is clearly independent of splicing in such disks.

To see that the above quantity is zero, we simply use Lemma 5.4, which shows that the relative Alexander grading is preserved by the change of diagrams maps. As such the top degree generator $\Theta^{\mathbf{w}} = \Theta^{\mathbf{z}}$ is preserved by the change of diagrams map (note that $\Theta^{\mathbf{w}} = \Theta^{\mathbf{z}}$ since there are exactly two basepoints per component). Since we defined the absolute Alexander grading of $\Theta^{\mathbf{w}}$ to be equal in two diagrams $(\Sigma, \alpha', \alpha)$ and $(\Sigma, \alpha'', \alpha)$, it follows that the above quantity is zero. The lemma statement now follows. \square

The analog of the previous lemma for two moves of the β -curves follows from the same argument. Similarly we can commute the maps induced by α -moves with the maps induced by β -moves.

Lemma 6.2. *Suppose that $(\Sigma, \alpha', \alpha, \beta, \beta', \mathbf{w}, \mathbf{z})$ is a Heegaard quadruple achieving weak admissibility, such that α' results from α by a sequence of handleslides and isotopies, and β' is obtained from β by a sequence of handleslides or isotopies. If $A \in \mathbb{A}(\Sigma, \alpha, \beta)$, then*

$$(\Phi_{\beta' \rightarrow \beta}^{\alpha \rightarrow \alpha'} \circ \Phi_{\beta \rightarrow \beta'}^{\alpha})(A) = (\Phi_{\beta \rightarrow \beta'}^{\alpha'} \circ \Phi_{\beta}^{\alpha \rightarrow \alpha'})(A).$$

Proof. This follows from the obvious adaptation of the proof of Lemma 6.1, using associativity on the level of homology classes. \square

Using the previous two lemmas, we can adapt the strategy of [JT12, Section 9] to show that if (Σ, α, β) and $(\Sigma, \alpha', \beta')$ are each admissible Heegaard diagrams for (Y, \mathbb{L}) with the same underlying Heegaard surface, such that α' is the result of a sequence of handleslides and isotopies amongst the α curves, and β' is the result of handleslides and isotopies amongst the β curves, then there are well defined maps

$$\Phi_{\beta \rightarrow \beta'}^{\alpha \rightarrow \alpha'} : \mathbb{A}(\Sigma, \alpha, \beta) \rightarrow \mathbb{A}(\Sigma, \alpha', \beta'),$$

which are functorial under composition, and also satisfy $\Phi_{\beta \rightarrow \beta}^{\alpha \rightarrow \alpha} = \text{id}$.

If the diagram \mathcal{H}_σ is the result of a $(1, 2)$ -stabilization of the diagram \mathcal{H} , and $A \in \mathbb{A}(\mathcal{H}, \pi, \mathfrak{s})$, then we can define a grading $\Phi_\sigma(A) \in \mathbb{A}(\mathcal{H}_\sigma, \pi, \mathfrak{s})$ by the formula

$$\Phi_\sigma(A)(\mathbf{x} \times c) = A(\mathbf{x}),$$

where c is the new intersection point on \mathcal{H}_σ .

Lemma 6.3. *If $(\Sigma, \alpha', \alpha, \beta)$ is an admissible triple associated to a sequence of α handleslides or isotopies, and $(\Sigma, \bar{\alpha}', \bar{\alpha}, \bar{\beta})$ is the triple resulting from performing $(1, 2)$ -stabilization to $(\Sigma, \alpha', \alpha, \beta)$, then we have*

$$(\Phi_\sigma \circ \Phi_{\beta}^{\alpha \rightarrow \alpha'})(A) = (\Phi_{\bar{\beta}}^{\bar{\alpha} \rightarrow \bar{\alpha}'} \circ \Phi_\sigma)(A).$$

Proof. To a homology triangle on $(\Sigma, \alpha', \alpha, \beta)$, we can pick a stabilized triangle on $(\Sigma, \bar{\alpha}', \bar{\alpha}, \bar{\beta})$, which has vertices at the new, higher degree intersection point of $\bar{\alpha}' \cap \bar{\alpha}$, as well as the two new intersection points of $\bar{\alpha}' \cap \bar{\beta}$ and $\bar{\alpha} \cap \bar{\beta}$. An easy computation using this homology class shows that the induced gradings satisfy the stated relation. \square

A similar lemma shows that the maps Φ_σ commute in the appropriate sense with handleslides and isotopies of the β curves.

Similarly we have the following:

Lemma 6.4. *If σ and σ' are two $(1, 2)$ -stabilizations, then*

$$(\Phi_\sigma \circ \Phi_{\sigma'})(A) = (\Phi_{\sigma'} \circ \Phi_\sigma)(A).$$

Proof. This is obvious from the formulas. \square

Additionally, to a diffeomorphism $\phi : (Y, \mathbb{L}) \rightarrow (Y, \mathbb{L})$ which maps L to L and is the identity on $\mathbf{w} \cup \mathbf{z}$, there is a tautological map

$$\phi_* : \mathbb{A}(\mathcal{H}, \pi, \mathfrak{s}) \rightarrow \mathbb{A}(\phi_* \mathcal{H}, \phi_* \pi, \phi_* \mathfrak{s}).$$

We have the following:

Lemma 6.5. *If $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ is a diagram for (Y, \mathbb{L}) and $\phi : (\Sigma, \mathbf{w} \cup \mathbf{z}) \rightarrow (\Sigma, \mathbf{w} \cup \mathbf{z})$ is a diffeomorphism which is isotopic to $\text{id}|_\Sigma$, relative $\mathbf{w} \cup \mathbf{z}$, then*

$$\Phi_{\alpha \rightarrow \phi_* \alpha}^{\beta \rightarrow \phi_* \beta} = \phi_*.$$

Proof. This follows from an adaptation of the proof of [JT12, Proposition 9.24] (except that we don't need to pay attention to whether triangles have holomorphic representatives). One decomposes ϕ into a composition of diffeomorphisms, such that for each, the diffeomorphism map can be defined as a composition of two small triangle maps (one for the α -equivalence, and one for a β -equivalence) (see the proof in [JT12] for more details). The small triangle map induces a map on gradings, which is equal by definition to the maps induced by α and β equivalences. \square

From [JT12], there is an important class of loops in the space of Heegaard diagrams, called simple handleswaps, and we need to check that there is no monodromy around such loops. The simple handleswap loop can be thought of as generating π_1 of the space of Heegaard diagrams. See [JT12, Definition 2.32] for a precise description. The move consists of one α -equivalence, one β -equivalence, and one diffeomorphism, all supported in a ball. We have the following, easier, version of [JT12, Proposition 9.25]:

Lemma 6.6. *If $H_1 \xrightarrow{e} H_2 \xrightarrow{f} H_3 \xrightarrow{g} H_1$ is a simple handleswap, then*

$$\Phi_g \circ \Phi_f \circ \Phi_e = \text{id},$$

as a map on $\mathbb{A}(\mathcal{H}, \pi, \mathfrak{s})$.

Proof. The maps Φ_e and Φ_f are maps induced by α - and β -handleslides. The map Φ_g is induced by a diffeomorphism. Since they are computed (in our context) by picking any homology triangle, we can simply pick the small triangle representatives. For such a choice of triangles, it is easy to verify the above identity. \square

Proposition 6.7. *For two admissible diagrams \mathcal{H} and \mathcal{H}' for (Y, \mathbb{L}) , there are maps $\Phi_{\mathcal{H} \rightarrow \mathcal{H}'} : \mathbb{A}(\mathcal{H}, \pi, \mathfrak{s}) \rightarrow \mathbb{A}(\mathcal{H}', \pi, \mathfrak{s})$, defined as a composition of the maps described above, for a choice of Heegaard moves from \mathcal{H} to \mathcal{H}' . The maps $\Phi_{\mathcal{H} \rightarrow \mathcal{H}'}$ are independent of the choice of intermediate Heegaard diagrams.*

Proof. We will show that the spaces $\mathbb{A}(\mathcal{H}, \pi, \mathfrak{s})$ define a “strong Heegaard invariant” from the set of diffeomorphism types of sutured manifolds which are link complements ([JT12, Definition 2.33]) to the category of affine sets over \mathbb{Q}^J . Axiom (1) is that we can separately define functorial maps associated to the subcategories of sutured manifolds corresponding to α -equivalences, β -equivalences, and diffeomorphisms. Axiom (2) states that the five distinguished rectangles commute. The first four are proven in Lemmas 6.1, 6.2, 6.3, and 6.4. The fifth states that the (1,2)-stabilization maps commute with diffeomorphism maps, which is obvious. Similarly, Axiom (3) states that the map induced by an isotopy of Σ is equal to map induced by α - and β -equivalence, which is verified in Lemma 6.5. Finally Axiom (4), handleswap invariance, is verified in Lemma 6.6. The result now follows from [JT12, Theorem 2.39]. \square

Lemma 6.8. *If $A \in \mathbb{A}(\mathcal{H}, \pi, \mathfrak{s})$, then the change of diagrams maps $\Phi_{\mathcal{H} \rightarrow \mathcal{H}'} : \mathcal{CFL}^\circ(\mathcal{H}, \mathfrak{s}) \rightarrow \mathcal{CFL}^\circ(\mathcal{H}', \mathfrak{s})$ are 0-graded maps with respect to the two gradings A and $\Phi_{\mathcal{H} \rightarrow \mathcal{H}'}(A)$.*

Proof. This is essentially a tautology from the definitions. One simply checks this for equivalences of the α - and β -curves, as well as (1,2)-stabilizations, and diffeomorphisms $\phi : (Y, \mathbb{L}) \rightarrow (Y, \mathbb{L})$. \square

6.2. Change of diagrams maps for Maslov gradings. We now sketch how to adapt the ideas of the previous section to define change of diagrams maps on the sets of Maslov gradings of complexes.

From our declaration of the absolute gradings for $\mathcal{CFL}^-(\mathcal{H}, \mathfrak{s})$ for a diagram $\mathcal{H} = (\Sigma, \alpha, \beta)$ for $((S^1 \times S^2)^{\#k}, \mathbb{U})$, when \mathbb{U} is an unlink with two basepoints per component, the absolute Maslov gradings $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ are determined by declaring the top degree $\text{gr}_{\mathbf{o}}$ generator of $\overline{HFL}(\Sigma, \alpha, \beta)$ to be in $\text{gr}_{\mathbf{o}}$ grading $\frac{1}{2}(k + |\mathbf{w}| - 1)$. By picking a simple diagram, it is easily computed that $\Theta_{\mathbf{w}} = \Theta_{\mathbf{z}}$ and $\text{gr}_{\mathbf{w}} = \text{gr}_{\mathbf{z}}$, on homology.

Recalling that $\mathbb{G}_{\mathbf{w}}(\mathcal{H}, \mathfrak{s})$ and $\mathbb{G}_{\mathbf{z}}(\mathcal{H}, \mathfrak{s})$ are defined as the sets of lifts of the relative $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ gradings on $\mathcal{CFL}^\circ(\mathcal{H}, \mathfrak{s})$, we note that if $(\Sigma, \alpha', \alpha, \beta)$ is an admissible triple, we can define a map

$$\Phi_{\alpha \rightarrow \alpha'}^\beta : \mathbb{G}_{\mathbf{o}}(\Sigma, \alpha, \beta) \rightarrow \mathbb{G}_{\mathbf{o}}(\Sigma, \alpha', \beta),$$

for $\mathbf{o} \in \{\mathbf{w}, \mathbf{z}\}$. If $G \in \mathbb{G}_{\mathbf{o}}(\Sigma, \alpha, \beta)$ we define $\Phi_{\alpha \rightarrow \alpha'}^\beta(G)$ by the formula

$$\Phi_{\alpha \rightarrow \alpha'}^\beta(G)(\mathbf{y}) = G(\mathbf{x}) + \text{gr}_{\mathbf{o}}(\theta) - \frac{1}{2}(k + |\mathbf{w}| - 1) - \mu(\psi) + 2n_{\mathbf{o}}(\psi),$$

for a homology triangle $\psi \in \pi_2(\theta, \mathbf{x}, \mathbf{y})$ with $\mathfrak{s}_{\mathbf{w}}(\psi)|_{Y_{\alpha\beta}} = \mathfrak{s}$ and $c_1(\mathfrak{s}_{\mathbf{w}}|_{Y_{\alpha'\alpha}})$ torsion. Note that the formula is invariant under splicing in disks on any of the ends, and hence is independent of the choice of intersection points \mathbf{x} and θ , as well as the triangle ψ . We can additionally define maps for moves of the β -curves, as well as diffeomorphisms and (1,2)-(de)stabilizations. The argument in the previous section goes through with only trivial modification to yield well defined, functorial change of diagrams maps $\Phi_{\mathcal{H} \rightarrow \mathcal{H}'} : \mathbb{G}_{\mathbf{o}}(\mathcal{H}, \mathfrak{s}) \rightarrow \mathbb{G}_{\mathbf{o}}(\mathcal{H}', \mathfrak{s})$.

Lemma 6.9. *If $G \in \mathbb{G}_{\mathbf{o}}(\mathcal{H}, \mathfrak{s})$, then the change of diagrams maps $\Phi_{\mathcal{H} \rightarrow \mathcal{H}'} : \mathcal{CFL}^\circ(\mathcal{H}, \mathfrak{s}) \rightarrow \mathcal{CFL}^\circ(\mathcal{H}', \mathfrak{s})$ are 0-graded maps with respect to the two gradings G and $\Phi_{\mathcal{H} \rightarrow \mathcal{H}'}(G)$.*

Proof. As in Lemma 6.8, this is essentially a tautology, and is checked easily for each elementary Heegaard move. \square

6.3. Coherent gradings. Using the previous two sections, we can define the notion of a **coherent grading**. Of course this notion is already implicit in Heegaard Floer homology, but we state it for convenience. Let $D(Y, \mathbb{L}, \mathfrak{s})$ denote the set of strongly \mathfrak{s} -admissible diagrams for (Y, \mathbb{L}) . We let $\vec{\mathbb{A}}(Y, \mathbb{L}, \pi, \mathfrak{s})$ denote the subset

$$\vec{\mathbb{A}}(Y, \mathbb{L}, \pi, \mathfrak{s}) \subseteq \prod_{\mathcal{H} \in D(Y, \mathbb{L}, \mathfrak{s})} \mathbb{A}(\mathcal{H}, \pi, \mathfrak{s})$$

consisting of tuples $(A_{\mathcal{H}})_{\mathcal{H} \in D(Y, \mathbb{L}, \mathfrak{s})}$ such that $\Phi_{\mathcal{H} \rightarrow \mathcal{H}'}(A_{\mathcal{H}}) = A_{\mathcal{H}'}$. We call such an element a **coherent Alexander grading**. We define the coherent Maslov gradings $\vec{\mathbb{G}}_{\mathbf{w}}(Y, \mathbb{L}, \mathfrak{s})$ and $\vec{\mathbb{G}}_{\mathbf{z}}(Y, \mathbb{L}, \mathfrak{s})$ analogously.

Note that there are canonical isomorphisms

$$\eta_{\mathcal{H}, \mathbb{A}} : \mathbb{A}(\mathcal{H}, \pi, \mathfrak{s}) \rightarrow \vec{\mathbb{A}}(Y, \mathbb{L}, \pi, \mathfrak{s}),$$

as well as similar isomorphisms $\eta_{\mathcal{H}, \mathbb{G}_{\mathbf{w}}}$ and $\eta_{\mathcal{H}, \mathbb{G}_{\mathbf{z}}}$ for the Maslov gradings. In particular, a choice of absolute Alexander or Maslov grading on a single diagram determines a coherent grading, under the map $\eta_{\mathcal{H}, \mathbb{A}}$.

Lemma 6.10. *The absolute Alexander and Maslov gradings we declared for unlinks in $(S^1 \times S^2)^{\#k}$ are coherent gradings.*

Proof. Let \mathcal{H} and \mathcal{H}' be two diagrams, and let $G_{\mathcal{H}}$ and $G_{\mathcal{H}'}$ be two gradings we declared in Section 5.4, for the diagrams \mathcal{H} and \mathcal{H}' (assume both $G_{\mathcal{H}}$ and $G_{\mathcal{H}'}$ are both lifts of the Alexander grading, or both lifts of $\text{gr}_{\mathbf{w}}$, or both lifts of $\text{gr}_{\mathbf{z}}$). Let $\Theta_{\mathcal{H}}^{\mathbf{w}} \in \widehat{HFL}(\mathcal{H}, \mathfrak{s}_0)$ denote the top $\text{gr}_{\mathbf{w}}$ -graded element. It is sufficient to show that

$$\left(\Phi_{\mathcal{H} \rightarrow \mathcal{H}'}(G_{\mathcal{H}}) \right) (\Theta_{\mathcal{H}'}^{\mathbf{w}}) = G_{\mathcal{H}'}(\Theta_{\mathcal{H}'}^{\mathbf{w}}).$$

By Lemma 5.4, the change of diagrams maps preserve the relative gradings, and as a consequence, $\Phi_{\mathcal{H} \rightarrow \mathcal{H}'}(\Theta_{\mathcal{H}}^{\mathbf{w}}) = \Theta_{\mathcal{H}'}^{\mathbf{w}}$. Combining this with Lemmas 6.8 and 6.9, we have

$$\left(\Phi_{\mathcal{H} \rightarrow \mathcal{H}'}(G_{\mathcal{H}}) \right) (\Theta_{\mathcal{H}'}^{\mathbf{w}}) = \left(\Phi_{\mathcal{H} \rightarrow \mathcal{H}'}(G_{\mathcal{H}}) \right) (\Phi_{\mathcal{H} \rightarrow \mathcal{H}'}(\Theta_{\mathcal{H}}^{\mathbf{w}})) = G_{\mathcal{H}}(\Theta_{\mathcal{H}}^{\mathbf{w}}),$$

which is equal by definition, to $G_{\mathcal{H}'}(\Theta_{\mathcal{H}'}^{\mathbf{w}})$. Hence $(\Phi_{\mathcal{H} \rightarrow \mathcal{H}'}(G_{\mathcal{H}}))(\Theta_{\mathcal{H}'}^{\mathbf{w}}) = G_{\mathcal{H}'}(\Theta_{\mathcal{H}'}^{\mathbf{w}})$, completing the proof. \square

7. ABSOLUTE ALEXANDER MULTI-GRADINGS

In this section, we describe a distinguished, coherent Alexander multi-grading on $CFL^{\circ}(Y, \mathbb{L}, \mathfrak{s})$ when L is J -null-homologous with respect to a grading assignment $\pi : C(L) \rightarrow J$.

Our approach is modeled on the construction of the absolute grading in [OS06]. Given a based link \mathbb{L} in a 3-manifold Y , we will pick a parametrized Kirby diagram $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ of (Y, \mathbb{L}) and take Heegaard triples representing surgery on \mathbb{S}_1 in $S^3 \setminus N(U)$ (where U is a collection of unknots) which represent $Y \setminus N(L)$. The Alexander grading is then defined using these Heegaard triples. Using the Kirby calculus argument from Section 4, we show that the grading is independent of the parametrized Kirby diagram and the Heegaard triple.

7.1. Construction of the absolute grading $A_{Y, \mathbb{L}, \mathbb{S}_1, \mathbb{P}}$. We now define the grading, in terms of a choice of parametrized Kirby diagram and some other auxiliary data. Suppose that (Y, \mathbb{L}) is a multi-based link in a three manifold, $\mathfrak{s} \in \text{Spin}^c(Y)$, and $\pi : C(L) \rightarrow J$ is a grading assignment, and that L is J -null-homologous. Suppose $S \subseteq Y$ is a J -Seifert surface, and λ is a choice of framing for L . Finally, suppose that $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ is a parametrized Kirby diagram of (Y, \mathbb{L}) .

We now pick a triple $\mathcal{T} = (\Sigma, \alpha, \beta, \gamma, \mathbf{w}, \mathbf{z})$ which is subordinate to a β -bouquet for \mathbb{S}_1 . By assumption $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ is a diagram for (S^3, \mathbb{U}) where $\mathbb{U} = (U, \mathbf{w}, \mathbf{z})$ is an unlink, and $(\Sigma, \alpha, \gamma, \mathbf{w}, \mathbf{z})$ is a diagram for (Y, \mathbb{L}) . Furthermore, the homeomorphism ϕ_0 in \mathbb{P} gives an identification between the components of \mathbb{L} and \mathbb{U} , so we naturally get a grading assignment $\pi \circ \phi_0 : C(U) \rightarrow J$, by using the grading assignment for \mathbb{L} .

The diagram $(\Sigma, \beta, \gamma, \mathbf{w}, \mathbf{z})$ represents an unlink, say $\mathbb{U}' = (U', \mathbf{w}, \mathbf{z})$, in $(S^1 \times S^2)^{\#k}$, though the number of components of \mathbb{U}' may be greater than \mathbb{U} , since each component of \mathbb{U}' has exactly two basepoints. Note however, that the two basepoints on a link component of \mathbb{U}' are on the same link component of the link \mathbb{L} in Y . Hence there is a natural map $C(U') \rightarrow C(L)$, which we can compose with the grading assignment for \mathbb{L} to get a grading assignment $\pi' : C(U') \rightarrow J$, hence allowing us to define an Alexander J -grading on $CFL^\circ((S^1 \times S^2)^{\#k}, \mathbb{U}', \mathfrak{s}_0)$ by collapsing the standard grading defined in Section 5.4.

Given a homology triangle $\psi \in \pi_2(\mathbf{x}, \theta, \mathbf{y})$ and a grading $j \in J$, we define

$$(13) \quad A_{\mathcal{T}}(\mathbf{y})_j = A_{S^3, \mathbb{U}}(\mathbf{x})_j + A_{(S^1 \times S^2)^{\#k}, \mathbb{U}'}(\theta)_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi)_j + \frac{\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi)), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2},$$

where $\widehat{\Sigma}_j$ is the closed surface in $W(S^3, \mathbb{S}_1)$ obtained by taking $\mathbb{U}_j \times [0, 1]$ in $W(S^3, L)$ and capping it off with any choice of Seifert surface for \mathbb{U}_j in S^3 , and $f^{-1}(S_j)$ in $S^3(\mathbb{S}_1)$. Here \mathbb{U}_j denotes the components of \mathbb{U}_j assigned grading j . Here θ is any intersection point, but it's worth noting that \mathbb{U}' has exactly two basepoints per component, so $\widehat{HFL}((S^1 \times S^2)^{\#k}, \mathbb{U}')$ is supported in a single Alexander grading. Since S^3 is an integer homology sphere, all choices of J -Seifert surfaces are homologous, so the above formula doesn't depend on the choice of Seifert surface in S^3 .

We then define $A_{Y, \mathbb{L}, S, \mathbb{P}} \in \vec{\mathbb{A}}(Y, \mathbb{L}, \pi, \mathfrak{s})$ to be the coherent grading

$$A_{Y, \mathbb{L}, S, \mathbb{P}} = \eta_{(\Sigma, \alpha, \gamma), \mathbb{A}}(A_{\mathcal{T}}),$$

where $\eta_{(\Sigma, \alpha, \gamma), \mathbb{A}} : \mathbb{A}((\Sigma, \alpha, \gamma), \pi, \mathfrak{s}) \rightarrow \vec{\mathbb{A}}(Y, \mathbb{L}, \pi, \mathfrak{s})$ is the inclusion map defined in the previous section. In the next section, we show that this is well defined.

Remark 7.1. It is easy to see that $\langle c_1(\mathfrak{s}), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}_j] \cdot [\widehat{\Sigma}_j]$ is always an even integer. As such, the j -component of the grading takes values in

$$\mathbb{Z} + \frac{1}{2}[\widehat{\Sigma} \setminus \widehat{\Sigma}_j] \cdot [\widehat{\Sigma}_j].$$

On the other hand, we note that it is easily seen that $[\widehat{\Sigma} \setminus \widehat{\Sigma}_j] \cdot [\widehat{\Sigma}_j] = \pm \ell k(L \setminus L_j, L_j)$, since the linking numbers of two null-homologous links are defined by picking a Seifert surface for one, and computing the intersection number of the other link with the Seifert surface.

7.2. Invariance of the absolute Alexander multi-grading. In this subsection, we prove invariance of the absolute Alexander multi-gradings from the Heegaard triple \mathcal{T} , subordinate to a bouquet of \mathbb{S}_1 , and the parametrized Kirby diagram \mathbb{P} . Invariance of the grading under Moves \mathcal{O}_1 and \mathcal{O}_2 follows from a similar argument to the one in [OS06]. Our argument deviates from [OS06] when we prove invariance from Move \mathcal{O}'_3 (and hence the choice of framing λ). Throughout this section, let $\pi : C(L) \rightarrow J$ be a fixed grading assignment and suppose that \mathbb{L} is J -null-homologous.

Lemma 7.2. *The coherent grading $A_{\mathcal{T}}$ is independent of isotopies of f (relative \mathbb{L}) or isotopies of \mathbb{S}_1 (relative \mathbb{U}).*

Proof. This is essentially a tautology. Isotopies of f , relative \mathbb{L} , for a fixed link $\mathbb{S}_1 \subseteq S^3_U$, result in an isotopy of the pullback of S under f . The topological quantities appearing at the end of the formula are unchanged. Thus an isotopy of f results in a grading which is related to the original grading by the change of diagrams map (on sets of gradings) associated to the isotopy. Isotopies of \mathbb{S}_1 are handled analogously. \square

Lemma 7.3. *For a fixed choice of β -bouquet for \mathbb{S}_1 , and a fixed Heegaard triple $\mathcal{T} = (\Sigma, \alpha, \beta, \gamma)$ subordinate to it, the grading $A_{\mathcal{T}}$ defined in Equation (13) defines an element of $\mathbb{A}(\Sigma, \alpha, \gamma, \pi, \mathfrak{s})$ which is independent of the choice of intersection points \mathbf{x} and θ , as well as the homology triangle $\psi \in \pi_2(\mathbf{x}, \theta, \mathbf{y})$.*

Proof. To see that $A_{\mathcal{T}} \in \mathbb{A}(\Sigma, \alpha, \gamma, \pi, \mathfrak{s})$, we note that if \mathbf{y} and \mathbf{y}' are two intersection points representing $\mathfrak{s} \in \text{Spin}^c(Y)$, then there is a disk $\phi \in \pi_2(\mathbf{y}', \mathbf{y})$, and one sees from the definition that $A_{\mathcal{T}}$ satisfies

$$A_{\mathcal{T}}(\mathbf{y}')_j - A_{\mathcal{T}}(\mathbf{y})_j = (n_{\mathbf{z}} - n_{\mathbf{w}})(\phi)_j = A(\mathbf{y}', \mathbf{y})_j.$$

We now show that $A_{\mathcal{T}}$ is independent of the triangle ψ and the choice of \mathbf{x} and θ . We first show that for fixed \mathbf{x} and θ , it is independent of ψ . If $\psi, \psi' \in \pi_2(\mathbf{x}, \theta, \mathbf{y})$ are two homology triangles which share the same endpoints, we can write

$$\psi' = \psi + \mathcal{P},$$

for a triply periodic domain \mathcal{P} . We have by [OS04b, Proposition 8.5], that

$$\mathfrak{s}_{\mathbf{w}}(\psi') = \mathfrak{s}_{\mathbf{w}}(\psi) + q_* PD[H(\mathcal{P})],$$

where $q_* : H^2(X_{\alpha\beta\gamma}, \partial X_{\alpha\beta\gamma}; \mathbb{Z}) \rightarrow H^2(X_{\alpha\beta\gamma}; \mathbb{Z})$ is the map in the long exact sequence of cohomology.

Let A denote the grading defined with the triangle ψ , and let A' denote the grading defined with ψ' . We now simply compute that

$$\begin{aligned} A'(\mathbf{y})_j &= A(\mathbf{x})_j + A(\theta)_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi')_j + \frac{\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi')), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2} \\ &= A(\mathbf{x})_j + A(\theta)_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi)_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\mathcal{P})_j + \frac{\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi)) + 2q_* PD[H(\mathcal{P})], [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2} \\ &= A(\mathbf{x})_j + A(\theta)_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi)_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\mathcal{P})_j + \langle q_* PD[H(\mathcal{P})], \widehat{\Sigma}_j \rangle + \frac{\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi)), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2} \\ &= A(\mathbf{y})_j \end{aligned}$$

since

$$\begin{aligned} (n_{\mathbf{w}} - n_{\mathbf{z}})(\mathcal{P})_j + \langle q_* PD[H(\mathcal{P})], \widehat{\Sigma}_j \rangle &= (n_{\mathbf{w}} - n_{\mathbf{z}})(\mathcal{P})_j + \langle PD[H(\mathcal{P})], \Sigma_j \rangle \\ &= (n_{\mathbf{w}} - n_{\mathbf{z}})(\mathcal{P})_j + \langle PD[H(\mathcal{P})], (\Sigma_{\alpha\beta\gamma})_j \rangle \\ &= 0 \end{aligned}$$

by Equation (2). \square

We now prove that the grading is invariant under the particular Heegaard triple subordinate to a given β -bouquet of \mathbb{S}_1 .

Lemma 7.4. *For a fixed β -bouquet \mathcal{B}^β of \mathbb{S}_1 , the coherent grading $\eta_{(\Sigma, \alpha, \gamma, \mathbb{A})}(A_{\mathcal{T}})$ on $CFL^\infty(Y, \mathbb{L}, \mathfrak{s})$ is independent from the Heegaard triple \mathcal{T} subordinate to \mathcal{B}^β .*

Proof. Any two triples subordinate to \mathcal{B}^β can be connected by the six moves of Lemma 3.6. Hence we need to show that if $\mathcal{T} = (\Sigma, \alpha, \beta, \gamma)$ and $\mathcal{T}' = (\Sigma', \alpha', \beta', \gamma')$ differ by a move in the above list, then

$$\Phi_{(\Sigma, \alpha, \gamma) \rightarrow (\Sigma', \alpha', \gamma')}(A_{\mathcal{T}}) = A_{\mathcal{T}}.$$

Consider Move (1). Supposing that α' differs from α by a sequence of handleslides or isotopies, we need to show that

$$A_{\mathcal{T}'} = \Phi_{\gamma}^{\alpha \rightarrow \alpha'}(A_{\mathcal{T}}).$$

By functoriality of the change of diagrams maps on gradings, we can assume that the quadruple $(\Sigma, \alpha', \alpha, \beta, \gamma)$ is admissible.

Suppose that $\psi_{\alpha\beta\gamma} \in \pi_2(\mathbf{x}_{\alpha\beta}, \theta_{\beta\gamma}, \mathbf{y}_{\alpha\gamma})$ and $\psi_{\alpha'\alpha\beta} \in \pi_2(\mathbf{y}_{\alpha'\alpha}, \theta_{\alpha\beta}, \mathbf{y})$ are homology triangles. Since $\text{Spin}^c(X_{\alpha'\beta\gamma}) \cong \text{Spin}^c(X_{\alpha\beta\gamma})$, we can write

$$\psi_{\alpha\beta\gamma} + \psi_{\alpha'\alpha\gamma} = \psi_{\alpha'\alpha\beta} + \psi_{\alpha'\beta\gamma}$$

for homology triangles $\psi_{\alpha'\alpha\beta} \in \pi_2(\theta_{\alpha'\alpha}, \mathbf{x}_{\alpha\beta}, \mathbf{x}_{\alpha'\beta})$ and $\psi_{\alpha'\beta\gamma} \in \pi_2(\mathbf{x}_{\alpha'\beta}, \theta_{\beta\gamma}, \mathbf{y})$ with $\mathfrak{s}_{\mathbf{w}}(\psi_{\alpha\beta\gamma}) = \mathfrak{s}_{\mathbf{w}}(\psi_{\alpha'\beta\gamma})$. By definition of the gradings, we have

$$\begin{aligned} A_{\mathcal{T}'}(\mathbf{y})_j - \Phi_{\gamma}^{\alpha \rightarrow \alpha'}(A_{\mathcal{T}, \mathfrak{s}})(\mathbf{y})_j &= A(\mathbf{x}_{\alpha'\beta})_j - A(\mathbf{x}_{\alpha\beta})_j - A(\theta_{\alpha'\alpha})_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi_{\alpha'\beta\gamma} - \psi_{\alpha\beta\gamma} - \psi_{\alpha'\alpha\gamma})_j \\ &= A(\mathbf{x}_{\alpha'\beta})_j - A(\mathbf{x}_{\alpha\beta})_j - A(\theta_{\alpha'\alpha})_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi_{\alpha'\alpha\beta})_j = A(\mathbf{x}_{\alpha'\beta})_j - \Phi_{\beta}^{\alpha \rightarrow \alpha'}(\mathbf{x}_{\alpha\beta})_j, \end{aligned}$$

which is zero since our declared absolute Alexander gradings on unlinks in $(S^1 \times S^2)^{\#k}$ are coherent by Lemma 6.10.

Independence from Moves (2), (4) and (5) follow in an identical fashion. Move (3) is an easy computation, analogous to Lemma 6.3, and Move (6) is a tautology. \square

We now consider dependence on the choice of β -bouquet \mathcal{B}^β for \mathbb{S}_1 :

Lemma 7.5. *The absolute grading is independent from the choice of bouquet for a fixed, framed one dimensional link $\mathbb{S}_1 \subseteq Y \setminus L$.*

Proof. This follows from the proof of [Juh16, Thm. 6.9]. We change the bouquet one arc at a time. A change of a single arc in the bouquet to a different arc can be achieved by performing a sequence of handleslides and isotopies amongst the β -curves, and amongst the γ -curves. Invariance under these moves is then achieved as in Lemma 7.4. \square

Lemma 7.6. *The absolute grading is independent from handleslides amongst the components of the framed link \mathbb{S}_1 . In particular, the absolute grading is invariant under move \mathcal{O}_1 .*

Proof. This follows again as in [OS06]. Handlesliding components of \mathbb{S}_1 across each other can be realized as handleslides of the β - and γ -curves across each other, and we've already shown invariance under those moves. \square

We now consider move \mathcal{O}_2 , corresponding to blowing up or blowing down along a ± 1 framed unknot K which is unlinked from \mathbb{S}_1 and U :

Lemma 7.7. *Suppose that K is an unknot in S^3 which is contained in a ball which is disjoint from \mathbb{S}_1 and U . Suppose that $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ is a parametrized Kirby diagram for (Y, \mathbb{L}) and let $\mathbb{P}' = (\phi_0, \lambda, \mathbb{S}_1 \cup \{K\}, f_K)$ denote the parametrized Kirby diagram obtained by adding K to \mathbb{S}_1 with framing ± 1 , and let f_K be the induced diffeomorphism. The gradings $A_{Y, \mathbb{L}, S, \mathbb{P}}$ and $A_{Y, \mathbb{L}, S, \mathbb{P}'}$ agree.*

Proof. If $\mathcal{T} = (\Sigma, \alpha, \beta, \gamma, \mathbf{w}, \mathbf{z})$ represents surgery on \mathbb{S}_1 , then we can form a triple representing surgery on $\mathbb{S}_1 \cup \{K\}$ by taking the connected sum of \mathcal{T} with the diagram shown in Figure 7.1.

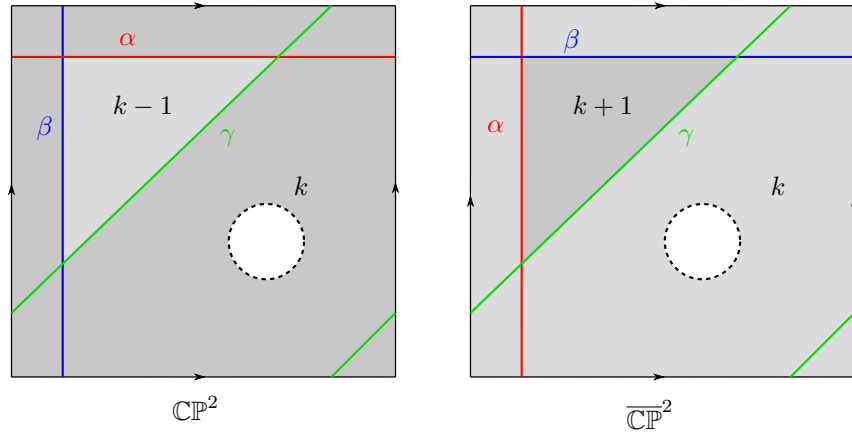


FIGURE 7.1. Taking connected sum of a surgery triple with one of the two triples shown above, at the dashed circle shown, results in a surgery triple for blowing up away from Σ . Multiplicities of a homology triangles are shown.

If ψ is a triangle on $(\Sigma, \alpha, \beta, \gamma)$, we can pick a homology triangle ψ' with the same multiplicity in the connect sum region and form $\psi \# \psi'$. We note that

$$(14) \quad \mathbf{s}_{\mathbf{w}}(\psi \# \psi') = \mathbf{s}_{\mathbf{w}}(\psi) \# \mathbf{s}'$$

for some \mathbf{s}' in $\text{Spin}^c(\mathbb{CP}^2)$ or $\text{Spin}^c(\overline{\mathbb{CP}^2})$. A diagram is shown in Figure 7.1, with a homology triangle drawn in. Using Equation (14)

$$\langle c_1(\mathbf{s}_{\mathbf{w}}(\psi \# \psi')), \widehat{\Sigma}_j \rangle = \langle c_1(\mathbf{s}_{\mathbf{w}}(\psi)), \widehat{\Sigma}_j \rangle$$

and that $[\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]$ is also unchanged. Since $(n_{\mathbf{w}} - n_{\mathbf{z}})(\psi)_j$ is also unchanged, the Alexander grading is unchanged and $A_{Y, \mathbb{L}, S, \mathbb{P}}$ and $A_{Y, \mathbb{L}, S, \mathbb{P}'}$ agree. \square

We will now show that the absolute gradings are invariant under move \mathcal{O}'_3 , for -1 framed unknots.

Lemma 7.8. *Suppose that $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ is a parametrized Kirby diagram for (Y, \mathbb{L}) and that K is an unknot which is unlinked from all components of \mathbb{S}_1 and U , except one component K of \mathbb{S}_1 , and to K it is linked as in move \mathcal{O}'_3 . Suppose K has framing -1 . Let $\mathbb{P}' = (\phi_0, \lambda', \mathbb{S}_1 \cup \{K\}, f_K)$ where f_K is the induced diffeomorphism, and λ' is the new framing on L . The gradings $A_{Y, \mathbb{L}, S, \mathbb{P}}$ and $A_{Y, \mathbb{L}, S, \mathbb{P}'}$ agree.*

Proof. A Heegaard triple for surgery on $\mathbb{S}_1 \cup \{K\}$ can be obtained by taking the connected sum of a diagram for surgery on \mathbb{S}_1 and the diagram shown in Figure 7.2, near a z -basepoint on the link component that K encircles (note the placement of the z -basepoint in the connected sum). Let $j \in J$ be the grading assigned to the basepoint z .

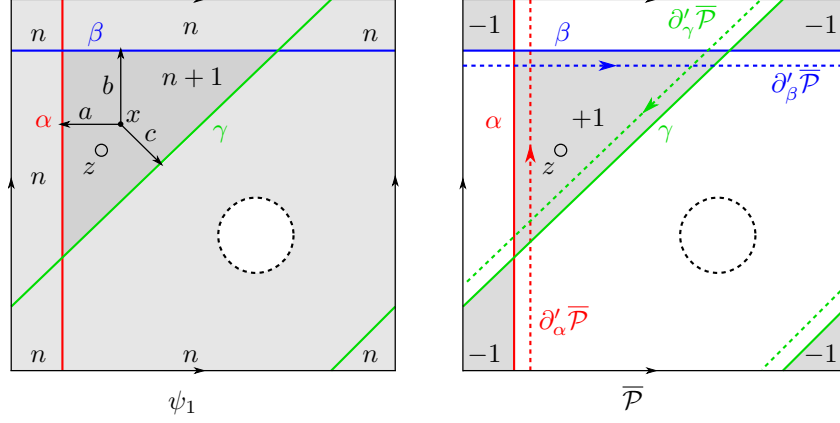


FIGURE 7.2. A diagram for surgery on $\mathbb{S}_1 \cup \{K\}$ where K has framing -1 . This is move \mathcal{O}'_3 . On the right is the triply periodic domain $\overline{\mathcal{P}}$ with $H(\overline{\mathcal{P}}) \in H_2(\overline{\mathbb{CP}}^2; \mathbb{Z})$ a generator. On the left is the homology triangle ψ_1 . On the left is also a dual spider with basepoint x and legs a, b and c . On the right are also the translates $\partial'_\alpha \overline{\mathcal{P}}, \partial'_\beta \overline{\mathcal{P}}$ and $\partial'_\gamma \overline{\mathcal{P}}$.

We note that there is a diffeomorphism between $W(S^3, \mathbb{S}'_1)$ and $W(S^3, \mathbb{S}_1) \# \overline{\mathbb{CP}}^2$ which is the identity outside a neighborhood of $B \times \{1\} \subseteq W(S^3, \mathbb{S}_1)$ for a ball $B \subseteq S^3$ containing K . Under this connected sum decomposition, we can describe $H_2(W(S^3, \mathbb{S}'_1); \mathbb{Z})$ as $H_2(W(S^3, \mathbb{S}_1); \mathbb{Z}) \oplus \mathbb{Z}$ where the copy of \mathbb{Z} is generated by an embedded sphere E in $W(S^3, \mathbb{S}'_1)$ formed by taking the obvious Seifert disk for K in $S^3 \times \{1\}$ and attaching the core of the 2-handle for K . Let Σ_j denote the surface $U \times [0, 1]$ in $W(S^3, \mathbb{S}_1)$ and let Σ'_j denote the analogous surface in $W(S^3, \mathbb{S}'_1)$. Under the identification of $W(S^3, \mathbb{S}'_1)$ as $W(S^3, \mathbb{S}_1) \# \overline{\mathbb{CP}}^2$ we now claim that the surface Σ'_j is equal to the connected sum of Σ_j (pushed outside of the connected sum region) and the sphere E . To see this, we note that we can push E radially out of the 2-handle attached along K , and give E the movie description shown in Figure 7.3, i.e. first an addition of a disk D_1 to the surface, then a 2-handle attached along K , then a disk D_2 attached to the surface. Inside of the 4-manifold $W(S^3, \mathbb{S}'_1)$, the sphere E is the union of D_1 and D_2 .

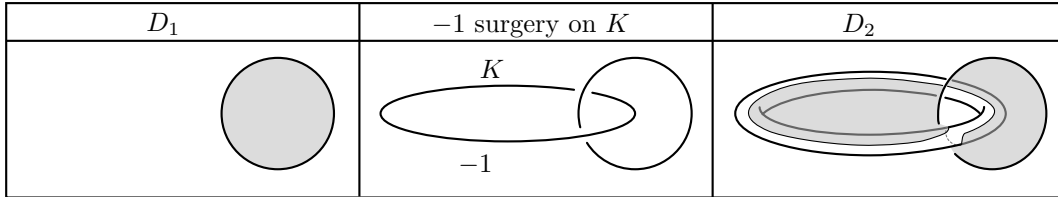


FIGURE 7.3. A movie presentation of the sphere E inside of $W(S^3, \mathbb{S}'_1)$. The disks D_1 and D_2 are shaded. On the right, we are showing only the complement of a neighborhood of K , inside of the surgered manifold $S^3(\mathbb{S}'_1)$. An annulus of D_2 is shown. The rest is D_2 compressing disk inside of the solid torus, which is not shown.

We can manipulate the movie for $\widehat{\Sigma}'_j$ by creating a 2-dimensional index 0/1 birth death which together cancel a small isotopy of the link, followed by surgery along K , then an additional 2-dimensional index 1/2 birth death followed by an isotopy of the link across the surgered solid torus. This is shown in Figure 7.4. We can move the band B_2 below the 2-handle attachment since they don't intersect, and we note that B_1 and B_2 are “dual bands”. This says that precisely Σ'_j is formed by taking Σ_j , pushed out of B (via the isotopies appearing as the first and last moves of the decomposition in Figure 7.4, and forming Σ'_j as the connected sum with the sphere E from Figure 7.3.

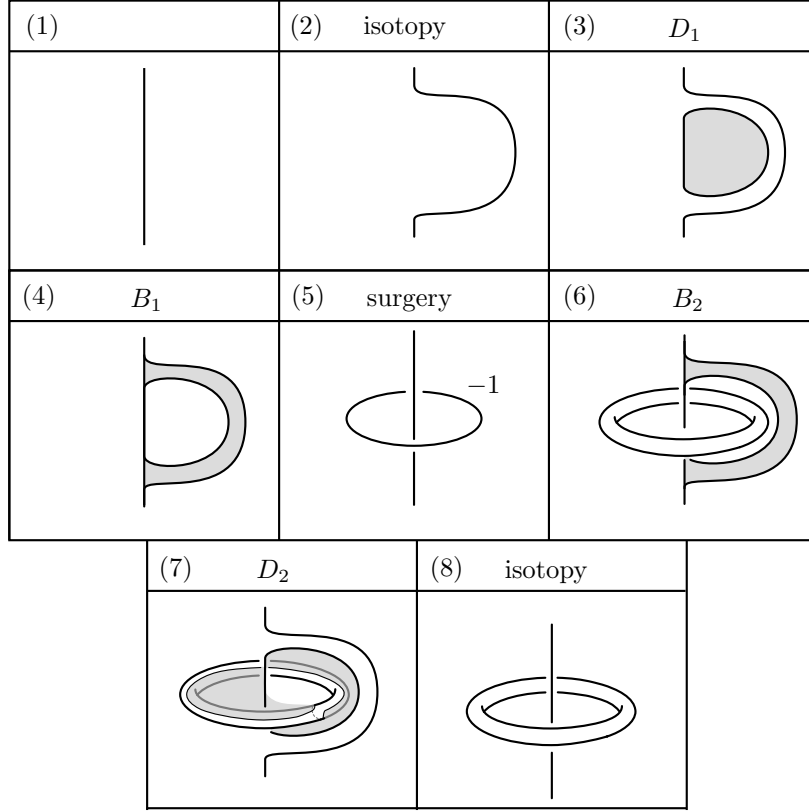


FIGURE 7.4. A movie for $\Sigma'_j \subseteq W(S^3, \mathbb{S}'_1)$, obtained by adding canceling 2-dimensional 0/1 and 1/2 critical points to Σ'_j . The shaded regions are the disks D_1 and D_2 , as well as the bands B_1 and B_2 . Moving the band B_2 below the 2-handle added along K , we get a description of Σ'_j inside of $W(S^3, \mathbb{S}_1) \# \overline{\mathbb{CP}}^2$ as $\Sigma_j \# E$, with Σ_j pushed out of the connected sum region. In the steps after -1 surgery on K , we are only showing $S^3 \setminus N(K) \subseteq S^3(K)$.

Note also that the homology element $H(\overline{\mathcal{P}})$ determined by the triply periodic domain $\overline{\mathcal{P}}$ can be seen to be $\pm E$, directly from its construction. The sign can be determined to be -1 since writing $\Sigma'_j = \Sigma_j \# aE$ and applying Equation (2) yields

$$1 = (n_{\mathbf{z}}(\overline{\mathcal{P}}) - n_{\mathbf{w}}(\overline{\mathcal{P}}))_j = \#(\Sigma'_j \cap H(\overline{\mathcal{P}})) = [\widehat{\Sigma}_j + aH(\overline{\mathcal{P}})] \cdot [H(\overline{\mathcal{P}})] = -a,$$

so $a = -1$.

Now letting ψ_1 be the triangle on the left of Figure 7.2, we compute $\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_1)), H(\overline{\mathcal{P}}) \rangle$ using [OS06, Prop. 6.3]. According to [OS06, Prop. 6.3], we have

$$\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_1)), H(\overline{\mathcal{P}}) \rangle = e(\overline{\mathcal{P}}) + \#(\partial \overline{\mathcal{P}}) - 2n_{\mathbf{w}}(\overline{\mathcal{P}}) + 2\sigma(\psi_1, \overline{\mathcal{P}}),$$

where e denotes the Euler measure, $\#\partial\overline{\mathcal{P}} = 3$ is the number of boundary components of $\overline{\mathcal{P}}$, and $\sigma(\psi_1, \overline{\mathcal{P}})$ is the “dual spider number”, a quantity defined in [OS06], by the formula

$$\sigma(\phi_1, \overline{\mathcal{P}}) = n_{\phi_1(x)}(\overline{\mathcal{P}}) + \#(a \cap \partial'_a \overline{\mathcal{P}}) + \#(b \cap \partial'_b \overline{\mathcal{P}}) + \#(c \cap \partial'_c \overline{\mathcal{P}}),$$

where a, b and c are arcs of the “dual spider”, and $\partial'_\tau \overline{\mathcal{P}}$ denotes the translation of the boundary component $\partial_\tau \overline{\mathcal{P}}$ of $\overline{\mathcal{P}}$, in the direction of the inward normal vector field, according to the periodic domain $\overline{\mathcal{P}}$. We compute now that $\sigma(\psi_1, \overline{\mathcal{P}}) = 1 - 1 - 1 - 1 = -2$. Adding the above terms up we get

$$\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_1)), H(\overline{\mathcal{P}}) \rangle = 3 - 4 = -1.$$

Now let ψ_k be the triangle defined by

$$\psi_k = \psi_1 + (k-1) \cdot \overline{\mathcal{P}}.$$

We have

$$\begin{aligned} \langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_k)), H(\overline{\mathcal{P}}) \rangle &= \langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_1)), H(\overline{\mathcal{P}}) \rangle + 2(k-1) \langle PD[H(\overline{\mathcal{P}})], H(\overline{\mathcal{P}}) \rangle \\ &= -1 - 2(k-1) = 1 - 2k. \end{aligned}$$

Notice also that $(n_{\mathbf{w}})_j$ is unchanged, but $(n_{\mathbf{z}})_j$ increases by $n_z(\psi_k) - n = k$ for ψ_k . Hence the j component of the Alexander grading changes by

$$\begin{aligned} &\frac{\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_k)), -H(\overline{\mathcal{P}}) \rangle - [-H(\overline{\mathcal{P}})] \cdot [-H(\overline{\mathcal{P}})]}{2} - (n_z(\psi_k) - n) \\ &= \frac{2k - 1 + 1}{2} - k = 0. \end{aligned}$$

In a similar way to Lemma 7.7, one sees that the other components of the Alexander grading are unchanged. \square

We now prove invariance under move \mathcal{O}'_3 in the case that the unknot K has framing $+1$. This is similar to the last lemma.

Lemma 7.9. *Suppose that $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ is a parametrized Kirby diagram for (Y, \mathbb{L}) and suppose that K is an unknot which is unlinked from all components of \mathbb{S}_1 and U , except one component of U , and it is linked as in move \mathcal{O}'_3 . Suppose K has framing $+1$. Let $\mathbb{P}' = (\phi_0, \lambda', \mathbb{S}_1 \cup \{K\}, f_K)$, where f_K is the induced diffeomorphism, and λ' is the new framing on L . The two gradings $A_{Y, \mathbb{L}, S, \mathbb{P}}$ and $A_{Y, \mathbb{L}, S, \mathbb{P}'}$ agree.*

Proof. The proof follows analogously to the proof of Lemma 7.8. Again let j denote the index of the grading that K is assigned to. Let ψ_{-1} and \mathcal{P} be the homology triangle and triply periodic domain shown in Figure 7.5.

Let Σ'_j denote the surface $U \times [0, 1]$ in $W(S^3, \mathbb{S}'_1)$ and let Σ_j denote the surface $U \times [0, 1]$ in $W(S^3, \mathbb{S}_1)$. As in Lemma 7.8, we can write $\Sigma'_j = \Sigma_j \# aH(\mathcal{P})$ if we identify $W(S^3, \mathbb{S}'_1)$ as $W(S^3, \mathbb{S}_1) \# \mathbb{CP}^2$. Arguing as before, we have now

$$1 = (n_{\mathbf{z}}(\mathcal{P}) - n_{\mathbf{w}}(\mathcal{P}))_j = \#(\Sigma'_j \cap H(\mathcal{P})) = a[H(\mathcal{P})] \cdot [H(\mathcal{P})] = a,$$

implying that $\Sigma'_j = \Sigma_j \# (+H(\mathcal{P}))$. Computing as before we have that

$$\sigma(\psi_{-1}, \mathcal{P}) = 1 - 1 - 1 - 1 = -2$$

$$\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_{-1})), H(\mathcal{P}) \rangle = 0 + 3 + 0 - 4 = -1.$$

Let ψ_k be the triangle defined by $\psi_k = \psi_{-1} + (k+1)\mathcal{P}$. We have

$$\begin{aligned} \langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_k)), H(\mathcal{P}) \rangle &= \langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_{-1})), H(\mathcal{P}) \rangle + 2(k+1) \langle PD[H(\mathcal{P})], H(\mathcal{P}) \rangle \\ &= 2k + 1. \end{aligned}$$

Hence blowing up along K and using the Spin^c structure $\mathfrak{s}_{\mathbf{w}}(\psi_k)$ thus produces a net change in the j component Alexander grading of

$$\frac{\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_k)), H(\mathcal{P}) \rangle - [H(\mathcal{P})] \cdot [H(\mathcal{P})]}{2} - (n_z(\psi_k) - n) = \frac{(2k+1) - 1}{2} - k = 0.$$

As in the previous two lemmas, the Alexander grading is unchanged in the components other than j . \square

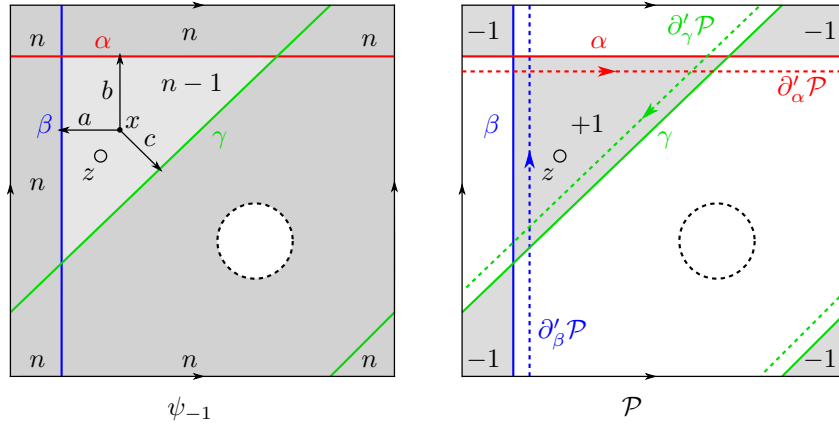


FIGURE 7.5. A diagram for surgery on $\mathbb{S}_1 \cup \{K\}$ where K has framing $+1$. This is move \mathcal{O}'_3 . Here \mathcal{P} is the periodic domain shown on the right which represents a generator $H(\mathcal{P}) \in H_2(\mathbb{CP}^2; \mathbb{Z})$. On the left is the homology triangle ψ_{-1} . We can assume that the diagram has a \mathbf{w} -basepoint in the same region as the dashed circle. On the left is also a dual spider with basepoint x and legs a, b and c . On the right are the translates $\partial'_\alpha \mathcal{P}, \partial'_\beta \mathcal{P}$ and $\partial'_\gamma \mathcal{P}$.

We now consider changing the identification ϕ_0 of \mathbb{U} with \mathbb{L} . Suppose that $\psi : (S^3, \mathbb{U}) \rightarrow (S^3, \mathbb{U})$ is an orientation preserving diffeomorphism (of 3-manifolds with multi-based links), and suppose that $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ is a parametrized Kirby diagram for (Y, \mathbb{L}) . Note that ψ determines a diffeomorphism $\psi^{\mathbb{S}_1} : S^3_U(\mathbb{S}_1) \rightarrow S^3_U(\psi(\mathbb{S}_1))$. Hence we can form the parametrization $\mathbb{P}' = (\phi_0, \lambda, \psi(\mathbb{S}_1), f \circ (\psi^{\mathbb{S}_1})^{-1})$.

Lemma 7.10. *Suppose that $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ is a parametrized Kirby diagram of (Y, \mathbb{L}) and that $\psi : (S^3, \mathbb{U}) \rightarrow (S^3, \mathbb{U})$ is an orientation preserving diffeomorphism (of 3-manifolds with multi-based links). Then $A_{Y, \mathbb{L}, S, \mathbb{P}}$ and $A_{Y, \mathbb{L}, S, \mathbb{P}'}$ agree, where $\mathbb{P}' = (\phi_0 \circ \psi, \lambda, \psi(\mathbb{S}_1), f \circ (\psi^{\mathbb{S}_1})^{-1})$.*

Proof. We take a triple $\mathcal{T} = (\Sigma, \alpha, \beta, \gamma)$ subordinate to a β -bouquet of \mathbb{S}_1 . The triple $\psi_* \mathcal{T} = (\psi_* \Sigma, \psi_* \alpha, \psi_* \beta, \psi_* \gamma)$ is subordinate to a β -bouquet for $\psi(\mathbb{S}_1)$. If $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ and $\psi_* \mathbf{y}$ denotes the corresponding intersection point on the surface $\psi_* \Sigma \subseteq S^3$, then

$$f_*(\mathbf{y}) = (f \circ (\psi^{\mathbb{S}_1})^{-1})_*(\psi_* \mathbf{y}).$$

Since the absolute gradings for unlinks in S^3 are determined by the multi-grading of the top $\text{gr}_{\mathbf{w}}$ -graded element, and ψ_* preserves the top graded element and also preserves the relative Alexander multi-grading, we know that $A(\psi_*(\theta))_j = A(\theta)_j$ for $\theta \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$, and similarly $A(\psi_*(\mathbf{x}))_j = A(\mathbf{x})_j$ for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Similarly, the value of $\frac{1}{2}(\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi)), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j])$ is tautologically unchanged. As such, the formulas for $A_{Y, \mathbb{L}, S, \mathbb{P}}$ and $A_{Y, \mathbb{L}, S, \mathbb{P}'}$ agree, so the gradings are the same. \square

Combining the results of this section, we can prove part (a) of Theorem 1.3:

Theorem 1.3(a). *For a fixed J -Seifert surface S of \mathbb{L} , the absolute gradings $A_{Y, \mathbb{L}, S, \mathbb{P}}$ depend only on the tuple (Y, \mathbb{L}, S) .*

Proof. By Lemma 7.3, for fixed \mathbb{P} , the gradings $A_{Y, \mathbb{L}, S, \mathbb{P}}$ depend at most on the Heegaard triple \mathcal{T} and \mathbb{P} . By Lemma 7.4 the grading is independent of the choice of Heegaard triple \mathcal{T} subordinate to a fixed bouquet for \mathbb{S}_1 . By Lemma 7.5, the grading is independent of the choice of bouquet subordinate to \mathbb{S}_1 . It follows that $A_{Y, \mathbb{L}, S, \mathbb{P}}$ depends on only Y, \mathbb{L}, S and \mathbb{P} . By Lemma 4.4, independence from \mathbb{P} will follow from showing independence of Moves $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}'_3$ and \mathcal{O}_4 . Invariance from \mathcal{O}_0 follows from Lemma 7.2. Invariance under Move \mathcal{O}_1 follows from Lemma 7.6. Invariance from Move \mathcal{O}_2 follows from Lemma 7.7. Invariance from Move \mathcal{O}'_3 follows from Lemmas 7.8 and 7.9. Finally, invariance from Move \mathcal{O}_4 follows from Lemma 7.10. \square

7.3. Gradings using α -bouquets. We defined the absolute grading using β -bouquets, but it will be useful to know that α -bouquets can be used as well. For definiteness, let us write $A_{Y,\mathbb{L},S}^\beta$ for the gradings defined in the previous section.

Repeating the above procedure, but using α -bouquets instead, yields an absolute grading $A_{Y,\mathbb{L},S}^\alpha$ on $CFL^\circ(Y, \mathbb{L}, \mathfrak{s})$, defined using α -bouquets.

Lemma 7.11. *The gradings $A_{Y,\mathbb{L},S}^\alpha$ and $A_{Y,\mathbb{L},S}^\beta$ are equal.*

Proof. We can think of this as a grading analog of [OS06, Lemma 5.2]. If \mathbb{S}_1 is a framed 1-dimensional link in $Y \setminus L$, and S' is a choice of J -Seifert surface in $Y(\mathbb{S}_1)$ for L , and $\mathfrak{s} \in \text{Spin}^c(W(Y, \mathbb{S}_1))$, we can define two maps, $F_{\mathbb{S}_1,S,S',\mathfrak{s}}^\alpha$ and $F_{\mathbb{S}_1,S,S',\mathfrak{s}}^\beta$ from $\vec{\mathbb{A}}(Y, \mathbb{L}, \pi, \mathfrak{s}|_Y)$ to $\vec{\mathbb{A}}(Y, \mathbb{L}, \pi, \mathfrak{s}|_{Y(\mathbb{S}_1)})$. Each is defined by taking a surgery diagram for a bouquet (either an α -bouquet or a β -bouquet, as appropriate) and then defining the grading $F_{\mathbb{S}_1,S,S',\mathfrak{s}}^\tau(A)$ by the formula

$$F_{\mathbb{S}_1,S,S',\mathfrak{s}}^\tau(A)(\mathbf{y})_j = A(\mathbf{x})_j + A_{(S^1 \times S^2)^{\#k}, \mathbb{U}}(\theta)_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi)_j + \frac{\langle c_1(\mathfrak{s}), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2}$$

where $\widehat{\Sigma}_j$ is formed by capping of the surface $L \times [0, 1] \subseteq W(Y, \mathbb{S}_1)$ with the J -Seifert surfaces S and S' , and ψ is a homology triangle with vertices on \mathbf{x}, θ and \mathbf{y} with $\mathfrak{s}_{\mathbf{w}}(\psi) = \mathfrak{s}$, and with θ representing the torsion Spin^c structure. As in the proof of invariance of the gradings, the maps $F_{\mathbb{S}_1,S,S',\mathfrak{s}}^\tau$ only depend on \mathfrak{s} through its restriction to the ends, by the same argument as in Lemma 7.3. Using the same associativity argument as in Lemma 6.1 (and noting that the homological quantities involving the capped J -Seifert surface are additive) we see that

$$F_{\mathbb{S}_1,S,S',\mathfrak{s}}^\alpha(A_{Y,\mathbb{L},S}^\alpha) = A_{Y(\mathbb{S}_1),\mathbb{L},S'}$$

and similarly for the β -gradings and β -maps. Essentially the same argument we used to show invariance of the gradings $A_{Y,\mathbb{L},S}^\tau$ now shows that the maps $F_{\mathbb{S}_1,S,S',\mathfrak{s}}^\tau$ yield well defined maps on coherent gradings.

Given a framed 2-sphere $\mathbb{S}_2 \subseteq Y$ and a Spin^c structure $\mathfrak{s} \in \text{Spin}^c(Y(\mathbb{S}_2))$, we can define a map on gradings $F_{\mathbb{S}_2}$, using the obvious formula, for diagrams which can be used to compute the 2-handle map. The same argument which shows that the 2-handle maps are well defined on the coherent chain complexes now shows that $F_{\mathbb{S}_2}$ yields a well defined map on coherent gradings. Furthermore, if S is a Seifert surface for \mathbb{L} which doesn't intersect \mathbb{S}_2 , then $F_{\mathbb{S}_2}(A_{Y,\mathbb{L},S}^\tau) = A_{Y(\mathbb{S}_2),\mathbb{L},S}^\tau$.

Note that if U is a 0-framed unknot contained in a ball in $Y \setminus L$, and \mathbb{S}_2 is the canonical framed 2-sphere in the surgered $Y_0(U)$ which cancels it, then the Seifert surface can be isotoped to miss \mathbb{S}_2 . Furthermore, the same computation as the one in [OS06, Lemma 14.6], except done only for homology triangles, shows that

$$F_{\mathbb{S}_2} \circ F_{U,S,\mathfrak{s}}^\tau = \text{id},$$

where \mathfrak{s} is the unique Spin^c structure which extends to the entire cobordism $W(Y, \mathbb{S}_1)(\mathbb{S}_2) \cong Y \times [0, 1]$, and which restricts to \mathfrak{t} on Y . This holds for both $\tau = \alpha$ and $\tau = \beta$. Using this, the argument now proceeds by straightforward modification of [OS06, Lemma 5.2]. \square

7.4. Dependence on the Seifert surface S . We now prove part (b) of Theorem 1.3:

Theorem 1.3(b). *If S and S' are two J -Seifert surfaces for a colored link \mathbb{L} in Y , then*

$$A_{Y,\mathbb{L},S'}(\mathbf{x})_j - A_{Y,\mathbb{L},S}(\mathbf{x})_j = \frac{\langle c_1(\mathfrak{s}), [S'_j \cup -S_j] \rangle}{2}.$$

In particular, if $c_1(\mathfrak{s})$ is torsion, then the absolute Alexander grading does not depend on the choice of J -Seifert surface.

Proof. Let S and S' be two choices of J -Seifert surfaces. Pick a parametrized Kirby diagram of (Y, \mathbb{L}) with framed 1-dimensional link \mathbb{S}_1 , and pick a Heegaard triple subordinate to a β -bouquet of \mathbb{S}_1 . Let $\widehat{\Sigma}$ denote the surface in the cobordism obtained by capping with S and let $\widehat{\Sigma}'$ be the surface obtained from capping with S' . Let $[F] = [S' \cup -S]$. As elements of $H_2(W(S^3, \mathbb{S}_1); \mathbb{Z})$, we have $[\widehat{\Sigma}'] = [\widehat{\Sigma}] + [F]$. We similarly define

$[F_j]$, $[\widehat{\Sigma}'_j]$ and $[\widehat{\Sigma}_j]$, and the analogous relation holds. Using the formula for the absolute grading, we have

$$\begin{aligned} A_{Y,\mathbb{L},S'}(\mathbf{x})_j - A_{Y,\mathbb{L},S}(\mathbf{x})_j &= \frac{\langle c_1(\mathfrak{s}), [\widehat{\Sigma}'_j] \rangle - \langle c_1(\mathfrak{s}), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}'] \cdot [\widehat{\Sigma}'_j] + [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2} \\ &= \frac{\langle c_1(\mathfrak{s}), [F_j] \rangle}{2} + \frac{-[F] \cdot [\widehat{\Sigma}_j] - [\widehat{\Sigma}] \cdot [F_j] - [F] \cdot [F_j]}{2} \\ &= \frac{\langle c_1(\mathfrak{s}), [F_j] \rangle}{2}, \end{aligned}$$

since $[F]$ and $[F_j]$ are in the image of the inclusion map $H_2(\partial W(S^3, \mathbb{S}_1); \mathbb{Z}) \rightarrow H_2(W(S^3, \mathbb{S}_1); \mathbb{Z})$, and hence the intersection number of either with anything in $H_2(W(S^3, \mathbb{S}_1); \mathbb{Z})$ vanishes. \square

If \mathfrak{s} is non-torsion, then in general the absolute Alexander grading depends on the choice of Seifert surface. In general, we can pick a diagram for a doubly based unlink $\mathbb{U} = (U, w, z)$ in Y by taking a diagram for (Y, w) and simply placing the z basepoint in the same region as w . The absolute Alexander grading on $CFL^\circ(Y, \mathbb{U}, \mathfrak{s})$ of $U_w^i V_z^j \cdot \mathbf{x}$ is given by just $j - i$, for the Seifert surface obtained by taking a path in surface between w and z which doesn't cross any α or β curves, and then pushing it off of Σ into both handlebodies U_α and U_β to get a disk with boundary U . The resulting Seifert surface of course depended on our choice of diagram and path in the surface, so in general one must be somewhat careful doing this. We illustrate this with an instructive example:

Example 7.12. Consider $Y = S^1 \times S^2$ with a doubly based unknot $\mathbb{U} = (U, w, z)$. In Figure 7.6, two diagrams \mathcal{H}_1 and \mathcal{H}_2 are shown. In each \mathcal{H}_i we have shown a path λ_i from w to z . We can specify two Seifert disks D_1 and D_2 , such that $D_i \cap \Sigma = \lambda_i$. Consider a Spin^c structure \mathfrak{s} represented by the intersection points in the diagram \mathcal{H}_1 . Note that

$$\langle c_1(\mathfrak{s}), H(P_i) \rangle = 2,$$

where P_1 and P_2 are the periodic domains on \mathcal{H}_1 and \mathcal{H}_2 (resp.) with multiplicity -1 in the bigon containing the two basepoints, and multiplicity $+1$ in the empty bigon. Note that $H(P_i) \in H_2(Y; \mathbb{Z})$ can be represented by isotopic spheres.

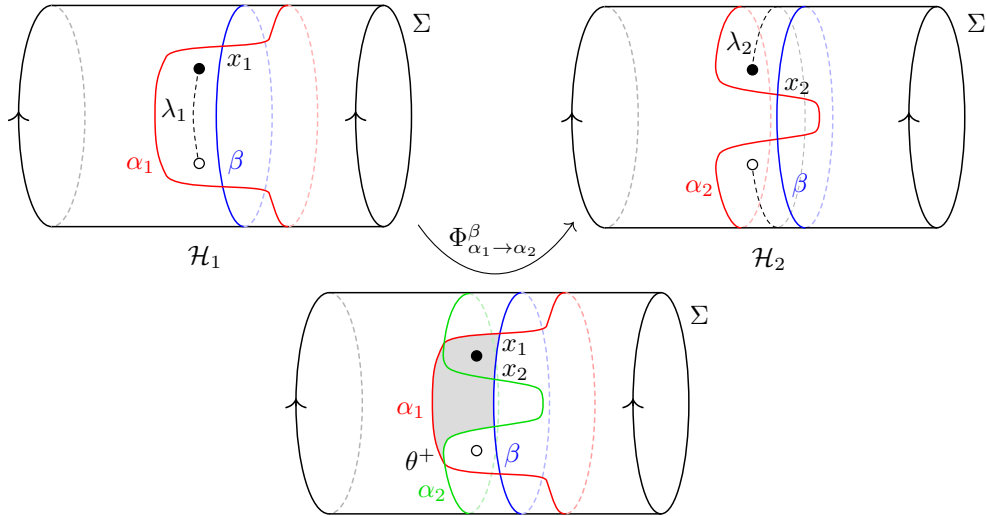


FIGURE 7.6. The two diagrams \mathcal{H}_1 and \mathcal{H}_2 for $(S^1 \times S^2, U, w, z)$ in a non-torsion grading, and the paths λ_1 and λ_2 . On the bottom is a Heegaard triple which can compute $\Phi_{\mathcal{H}_1 \rightarrow \mathcal{H}_2}$. The disks D_1 and D_2 intersect Σ along λ_1 and λ_2 . Shaded is a the domain of a homology triangle which can be used to compute $\Phi_{\mathcal{H}_1 \rightarrow \mathcal{H}_2}(A_{D_1})$.

Let D_1 denote the disk constructed from λ_1 , and let D_2 denote the disk constructed using λ_2 . Using our orientation conventions, it is not hard to see that

$$D_2 \cup (-D_1) = -H(P_i).$$

To compute the signs, it's easiest to use Equation (7) to compute the intersection number of $H(P_i)$ with a knot of the form $S^1 \times \{pt\}$. By drawing a picture of the Seifert disks, and using our orientation conventions, it's easy to see that $D_2 \cup (-D_1)$ intersects the same knot with the opposite orientation. Hence

$$\langle c_1(\mathfrak{s}), D_2 \cup (-D_1) \rangle = -2.$$

On both diagrams \mathcal{H}_i , the absolute Alexander grading with Seifert surface D_i is given by $A_{D_i}(U^i V^j \cdot \mathbf{x}) = j - i$ for any intersection point x . We can move from \mathcal{H}_1 to \mathcal{H}_2 by a single α move, shown in Figure 7.6. We now explicitly compute that, as gradings on \mathcal{H}_2

$$\Phi_{\mathcal{H}_1 \rightarrow \mathcal{H}_2}(A_{D_1})(x_2) = A_{D_1}(x_1) + (n_w - n_z)(\psi) = A_{D_1}(x_1) + 1.$$

Since $A_{D_i}(x_i) = 0$ for both $i = 1$ and $i = 2$, in terms of coherent gradings this yields

$$A_{D_2} = A_{D_1} - 1 = A_{D_1} + \frac{\langle c_1(\mathfrak{s}), D_2 \cup (-D_1) \rangle}{2},$$

agreeing with the previous theorem.

8. ABSOLUTE MASLOV GRADINGS

In this section, we prove parts (c) and (d) of Theorem 1.3, and describe the two absolute Maslov gradings, $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$, on $\mathcal{CFL}^\circ(Y, \mathbb{L}, \mathfrak{s})$. The grading $\text{gr}_{\mathbf{w}}$ is defined when \mathfrak{s} is torsion, and $\text{gr}_{\mathbf{z}}$ is defined when $\mathfrak{s} - PD[L]$ is torsion.

The construction of the absolute Maslov gradings $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ mimics the construction from [OS06]. To simplify the construction from [OS06], we first need to state a formula due to Sarkar for the change of the Maslov index due to adding a triple periodic domain to a homology triangle.

Lemma 8.1. *Suppose that \mathcal{P} is triply periodic domain and ψ is a homology triangle, on the Heegaard triple $(\Sigma, \alpha, \beta, \gamma, \mathbf{w}, \mathbf{z})$. Then*

$$\mu(\psi + \mathcal{P}) - \mu(\psi) = 2n_{\mathbf{w}}(\mathcal{P}) + \frac{c_1(\mathfrak{s}_{\mathbf{w}}(\psi + \mathcal{P}))^2 - c_1(\mathfrak{s}_{\mathbf{w}}(\psi))^2}{4}.$$

The proof can be found in [Sar11a, Section 5.1].

We now describe the absolute Maslov gradings. The $\text{gr}_{\mathbf{w}}$ -grading will be defined on intersection points \mathbf{y} with $\mathfrak{s}_{\mathbf{w}}(\mathbf{y})$ torsion, and the $\text{gr}_{\mathbf{z}}$ -grading will be defined on intersection points with $\mathfrak{s}_{\mathbf{z}}(\mathbf{y})$ torsion. For a link where the sum of the link components is not rationally null-homologous, we may be able to define one of these, but not the other (see Example 5.5).

We pick a parametrized Kirby diagram $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ of (Y, \mathbb{L}) , and a triple $(\Sigma, \alpha, \beta, \gamma)$ which is subordinate to a β -bouquet for \mathbb{S}_1 . By definition $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ represents a collection of unlinks in $(S^1 \times S^2)^{\#k}$ for some k . In Section 5.4, we defined two absolute gradings, $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ on $\widehat{\mathcal{CFL}}((S^1 \times S^2)^{\#k}, \mathbb{U}, \mathfrak{s}_0)$, for an unlink \mathbb{U} with various configurations of basepoints. If $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ and $\psi \in \pi_2(\mathbf{x}, \theta, \mathbf{y})$ is a homology triangle, we define

$$\text{gr}_{\mathbf{w}}(\mathbf{y}) = \text{gr}_{\mathbf{w}}(\mathbf{x}) + \text{gr}_{\mathbf{w}}(\theta) - \frac{1}{2}(k + |\mathbf{w}| - 1) - \mu(\psi) + 2n_{\mathbf{w}}(\psi) + \frac{c_1(\mathfrak{s}_{\mathbf{w}}(\psi))^2 - 2\chi(W) - 3\sigma(W)}{4},$$

where $W = W(S^3, \mathbb{S}_1)$. Similarly we define

$$\text{gr}_{\mathbf{z}}(\mathbf{y}) = \text{gr}_{\mathbf{z}}(\mathbf{x}) + \text{gr}_{\mathbf{z}}(\theta) - \frac{1}{2}(k + |\mathbf{w}| - 1) - \mu(\psi) + 2n_{\mathbf{z}}(\psi) + \frac{c_1(\mathfrak{s}_{\mathbf{z}}(\psi))^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

Here θ is an intersection point which represents the torsion Spin^c structure on (Σ, β, γ) .

We can now prove part (c) of Theorem 1.3.

Theorem 1.3(c). *The absolute gradings $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$, listed above, are independent of the intersection points \mathbf{y} and \mathbf{x} , the homology triangle ψ , as well as the parametrized Kirby diagram $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$.*

Proof. Most of the proof proceeds as in the proof of Theorem 1.3(a). The main difference is in the proof that the formulas are independent of the homology triangle ψ and intersection points. Independence of the absolute grading from \mathbf{x} and θ can be proven by splicing in disks on those ends and seeing that the formula doesn't change. To see more generally that the formula is invariant under the choice of ψ , we note that any two homology triangles with the same endpoints differ by a triply periodic domain. Suppose that ψ and $\psi + \mathcal{P}$ are two such triangles differing by a triply periodic domain \mathcal{P} . Letting $\text{gr}_{\mathbf{w}}^{\psi}(\mathbf{y})$ and $\text{gr}_{\mathbf{w}}^{\psi+\mathcal{P}}(\mathbf{y})$ denote the formulas, computed with ψ or $\psi + \mathcal{P}$, respectively, we observe that the difference is

$$\text{gr}_{\mathbf{w}}^{\psi+\mathcal{P}}(\mathbf{y}) - \text{gr}_{\mathbf{w}}^{\psi}(\mathbf{y}) = -\mu(\psi + \mathcal{P}) + \mu(\psi) + 2n_{\mathbf{w}}(\mathcal{P}) + \frac{c_1(\mathfrak{s}_{\mathbf{w}}(\psi + \mathcal{P}))^2 - c_1(\mathfrak{s}_{\mathbf{w}}(\psi))^2}{4},$$

which is zero by Sarkar's formula from Lemma 8.1. Analogously, using the computations from the proof of invariance of the Alexander gradings in the previous subsection, invariance from the moves of Lemma 4.4 follows similarly. The same argument works for showing invariance of $\text{gr}_{\mathbf{z}}$. \square

The following is part (d) of Theorem 1.3:

Theorem 1.3(d). *The absolute Maslov and collapsed Alexander grading satisfy*

$$A = \frac{1}{2}(\text{gr}_{\mathbf{w}} - \text{gr}_{\mathbf{z}}).$$

Proof. We pick a parametrized Kirby diagram and associated surgery triple for (Y, \mathbb{L}) . We compute $\frac{1}{2}$ times the difference between the expressions defining $\text{gr}_{\mathbf{w}}(\mathbf{x})$ and $\text{gr}_{\mathbf{z}}(\mathbf{x})$. We note that by Equation (11), the formula $A = \frac{1}{2}(\text{gr}_{\mathbf{w}} - \text{gr}_{\mathbf{z}})$ holds for the gradings associated to unlinks in $(S^1 \times S^2)^{\#k}$. By Lemma 3.3, we have that $\mathfrak{s}_{\mathbf{w}}(\psi) - \mathfrak{s}_{\mathbf{z}}(\psi) = PD[\Sigma_{\alpha\beta\gamma}]$. Under the inclusion $X_{\alpha\beta\gamma} \hookrightarrow W(Y, \mathbb{S}_1)$, we have that $PD[\Sigma_{\alpha\beta\gamma}] = PD[\Sigma]$. Noting that

$$\frac{c_1(\mathfrak{s}_{\mathbf{w}}(\psi))^2 - c_1(\mathfrak{s}_{\mathbf{z}}(\psi))^2}{4} = c_1(\mathfrak{s}_{\mathbf{w}}(\psi)) \cup PD[\Sigma] - PD[\Sigma] \cup PD[\Sigma] = \langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi)), [\widehat{\Sigma}] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}],$$

we see that the expression defining $\frac{1}{2}(\text{gr}_{\mathbf{w}}(\mathbf{x}) - \text{gr}_{\mathbf{z}}(\mathbf{x}))$ becomes exactly the expression defining $A(\mathbf{x})$. \square

Remark 8.2. Note that if L is null-homologous and \mathfrak{s} is torsion, if we tensor $\mathcal{CFL}^{\circ}(Y, \mathbb{L}, \mathfrak{s})$ with $\mathbb{Z}_2[U_{\mathbf{w}}, V_{\mathbf{z}}]/(V_{\mathbf{z}} - 1)$ or $\mathbb{Z}_2[U_{\mathbf{w}}, V_{\mathbf{z}}]/(U_{\mathbf{w}} - 1)$, the gradings $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ both give the standard absolute gradings on $\widehat{CF}(Y, \mathbf{w}, \mathfrak{s})$ or $\widehat{CF}(Y, \mathbf{z}, \mathfrak{s})$ (respectively), since a parametrized Kirby diagram of a link naturally gives a parametrized Kirby diagram of the 3-manifold, by forgetting about half of the basepoints. An alternate way to compute the collapsed collapsed Alexander grading is then to just compute the two absolute gradings on $\widehat{CF}(Y, \mathbf{w}, \mathfrak{s})$ or $\widehat{CF}(Y, \mathbf{z}, \mathfrak{s})$, and then define $A = \frac{1}{2}(\text{gr}_{\mathbf{w}} - \text{gr}_{\mathbf{z}})$, on intersection points.

Remark 8.3. The absolute gradings $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ we've written here are normalized so that $HF^-(S^3)$ has a top degree generator in grading 0. In [OS06] the gradings are normalized so that $HF^-(S^3)$ has top degree generator in grading -2 . Similarly the gradings in [HMZ] are normalized so that HF^- of disjoint unions of S^3 have top degree generator in grading -2 . If one wants to go from one convention to the other, one simply subtracts 2 from our Maslov gradings to get ones consistent with the other theories. Hopefully this won't cause confusion.

9. LINK COBORDISMS AND GRADING CHANGE FORMULAS

In this section we discuss the grading change for link cobordisms. We first consider the maps associated to 4-dimensional handles added away from the link, then we consider the maps associated to the surface (band maps, quasi-stabilization, and disk-stabilization) separately, and then prove that the formulas are additive to obtain the stated grading change formulas in general.

9.1. Grading changes of maps associated to elementary link cobordisms. In this subsection, we compute the Alexander grading change for the constituent maps appearing in the link cobordism maps. We first consider the disk-stabilization maps:

Lemma 9.1. *The disk stabilization maps $S_{\mathbb{U}, D}^{\pm}$ satisfy $A_{Y, \mathbb{L} \cup \mathbb{U}, S \cup D}(S_{\mathbb{U}, D}^{\pm}(\mathbf{x}))_j - A_{Y, \mathbb{L}, S}(\mathbf{x})_j = 0$. Similarly the maps $S_{\mathbb{U}, D}^{\pm}$ both induce $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ grading changes of $+\frac{1}{2}$.*

Proof. First consider the Alexander grading change. We take a parametrized Kirby diagram and a Heegaard triple $\mathcal{T} = (\Sigma, \alpha, \beta, \gamma, \mathbf{w}, \mathbf{z})$ representing surgery on the associated link \mathbb{S}_1 in $S^3 \setminus U$, which results in (Y, \mathbb{L}) , with the additional assumption that $w, z \in \Sigma$ and $D \cap \Sigma$ in an arc from w to z which is disjoint from $(\alpha \cup \beta \cup \gamma \cup \mathbf{w} \cup \mathbf{z})$. We then form the triple $\overline{\mathcal{T}}$ by disk-stabilizing \mathcal{T} by adding in isotopic α_0, β_0 and γ_0 curves which each bound a ball containing $D \cap \Sigma$. The triple $\overline{\mathcal{T}}$ represents surgery on the same framed link \mathbb{S}_1 in S^3 , but now the link \mathbb{L} has an extra unknot component U . By picking \mathbb{S}_1 so that it doesn't intersect D , we view D as being embedded in S^3 , $(S^1 \times S^2)^{\#k}$ and Y . Pick a triangle $\psi_{\alpha\beta\gamma} \in \pi_2(\mathbf{x}_{\alpha\beta}, \theta_{\beta\gamma}, \mathbf{x}_{\alpha\gamma})$ and a stabilization $\overline{\psi} \in \pi_2(\mathbf{x}_{\alpha\beta} \times \theta^+, \theta_{\beta\gamma} \times \theta^+, \mathbf{x}_{\alpha\gamma} \times \theta^+)$. The surface $\widehat{\Sigma}_j$ appearing in the link cobordism associated to \mathcal{T} is obtained from $\widehat{\Sigma}_j$ by adding a null-homologous sphere, and hence the homological quantities in the grading formulas are unchanged. An easy computation using triangles ψ and $\overline{\psi}$ as above shows that

$$A_{\overline{\mathcal{T}}}(\mathbf{x}_{\alpha\gamma} \times \theta^+)_j - A_{\mathcal{T}}(\mathbf{x}_{\alpha\gamma})_j = A(\mathbf{x}_{\alpha\beta} \times \theta^+)_j - A(\mathbf{x}_{\alpha\beta})_j + A(\theta_{\beta\gamma} \times \theta^+)_j - A(\theta_{\beta\gamma})_j,$$

and hence it is sufficient to show that the maps $S_{\mathbb{U},D}^{\pm}$ have grading change zero for unlinks in S^3 , which is obvious from the definition of the Alexander grading on unlinks in $(S^1 \times S^2)^{\#k}$.

We now consider the Maslov grading changes. Adapting the above argument, by stabilizing the homology triangle by adding in a small triangle with Maslov index 0 which crosses over none of the basepoints, we need only consider the grading change associated to performing the disk stabilization maps to an unlink in S^3 . It follows immediately from the formula that $S_{\mathbb{U},D}^+$ and $S_{\mathbb{U},D}^-$ are both $+\frac{1}{2}$ graded with respect to both $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$. \square

We now prove the analogous result for the quasi-stabilization maps. We recall that the quasi-stabilization maps $S_{w,z}^{\pm}$ and $T_{w,z}^{\pm}$ were defined in [Zem16b] and were associated to the decorated link cobordisms shown in Figure 9.1.

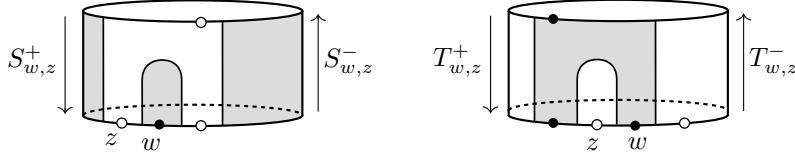


FIGURE 9.1. The decorated link cobordisms associated to the quasi-stabilization maps $S_{w,z}^{\pm}$ and $T_{w,z}^{\pm}$. Shown are the decorated surfaces $L \times [0, 1]$, which sit inside of $Y \times [0, 1]$.

Lemma 9.2. *The quasi-stabilization maps $S_{w,z}^{\pm}$ induce Alexander grading change $+\frac{1}{2}\delta(\pi(K), j)$ in the j component of the Alexander multi-grading, where K is the component of L containing w and z . Similarly the maps $T_{w,z}^{\pm}$ induce an Alexander grading change of $-\frac{1}{2}\delta(\pi(K), j)$.*

The maps $S_{w,z}^{\pm}$ also induce a $+\frac{1}{2}$ grading change of $\text{gr}_{\mathbf{w}}$ and a $-\frac{1}{2}$ grading change of $\text{gr}_{\mathbf{z}}$. Analogously $T_{w,z}^{\pm}$ induces a $-\frac{1}{2}$ grading change of $\text{gr}_{\mathbf{w}}$ and a $+\frac{1}{2}$ grading change of $\text{gr}_{\mathbf{z}}$.

Proof. This follows analogously to the proof of Lemma 9.1. Consider first the maps $S_{w,z}^{\pm}$. We take a surgery triple $\mathcal{T} = (\Sigma, \alpha, \beta, \gamma)$ and its quasi-stabilization along an α_s curve (we add w and z , the α_s curve is allowed to travel around the diagram, but we add two curves β_0 and γ_0 which both bound disks in Σ containing w and z and no other basepoints or curves). We note that the diagram (Σ, β, γ) becomes disk-stabilized, not quasi-stabilized. If $\mathbf{x}_{\alpha\beta}, \mathbf{x}_{\alpha\gamma}$ and $\theta_{\beta\gamma}$ are intersection points, as in the previous lemma, we have that

$$A_{\overline{\mathcal{T}}}(\mathbf{x}_{\alpha\gamma} \times \theta_{\mathbf{w}}^+)_j - A_{\mathcal{T}}(\mathbf{x}_{\alpha\gamma})_j = A(\mathbf{x}_{\alpha\beta} \times \theta_{\mathbf{w}}^+)_j - A(\mathbf{x}_{\alpha\beta})_j + A(\theta_{\beta\gamma} \times \theta^+)_j - A(\theta_{\beta\gamma})_j.$$

Now $A(\mathbf{x}_{\alpha\beta} \times \theta_{\mathbf{w}}^+)_j - A(\mathbf{x}_{\alpha\beta})_j$ is equal to the grading induced by the quasi-stabilization map $S_{w,z}^+$ on unlinks in S^3 , which is easily seen to be $+\frac{1}{2}$ by definition of the absolute grading on those links. Now $A(\theta_{\beta\gamma} \times \theta^+)_j - A(\theta_{\beta\gamma})_j$ is equal to the grading change of disk stabilization on unlinks in $(S^1 \times S^2)^{\#k}$, which is zero by the definition of the absolute grading on unlinks in $(S^1 \times S^2)^{\#k}$. The grading change for $T_{w,z}^{\pm}$, as well as the Maslov gradings $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ are described analogously. \square

If B is a band for a link L , with its ends on components of L of the same grading, with J -Seifert surface S , then we naturally get an J -Seifert surface $S \cup B$ for $L(B)$. Recall that a band B for L , intersects L only along the boundary of B . Note that we defined a J -Seifert surface to be an immersed surface, not necessarily an embedded surface, so it isn't important that B doesn't intersect S , since we only need a homology class defined by $S \cup B$ to compute the grading. In general, there is no hope of assuming that a band B occurs in the complement of the Seifert surfaces. Nonetheless, if B is a with ends on link components with grading assignment j and B intersects $S_j = \pi^{-1}(j) \subseteq S$ in a collection of arcs and closed 1-manifolds, it is easy to surger $S_j \cup B$ to make it an embedded surface without changing the homology class. This of course is only allowable if the intersection of B is with S_j and B is also attached to link components with the grading assignment j .

Lemma 9.3. *Suppose (Y, \mathbb{L}) is a 3-manifold with multi-based link and J -Seifert surface S , and B is a band for \mathbb{L} , between link components with the same grading assignment, j_0 , then*

$$A_{Y, \mathbb{L}(B), S \cup B}(F_B^{\mathbf{z}}(\mathbf{x}))_j - A_{Y, \mathbb{L}, S}(\mathbf{x})_j = +\frac{1}{2}\delta(j, j_0),$$

and

$$A_{Y, \mathbb{L}(B), S \cup B}(F_B^{\mathbf{w}}(\mathbf{x}))_j - A_{Y, \mathbb{L}, S}(\mathbf{x})_j = -\frac{1}{2}\delta(j, j_0).$$

Proof. We pick a parametrized Kirby diagram $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ which is nicely adapted to the band, as we now describe. Let $\mathbb{U} \subseteq S^3$ be a multi-based unlink and let $\phi_0 : \mathbb{U} \rightarrow \mathbb{L}$ be a diffeomorphism of multi-based, oriented 1-manifolds. We pick a band B_0 in S^3 between the two components of \mathbb{U} which correspond under ϕ_0 to the components connected by B , and that the configuration of basepoints with respect to the bands is the same for \mathbb{U} and B_0 and \mathbb{L} and B . We can assume that B_0 is a “standard” band for \mathbb{U} , i.e. we can assume that all components of \mathbb{U} can be put on an embedded sphere, and the band B_0 also lies on that sphere.

We can add sutures to $\partial N(\mathbb{U} \cup B_0)$ and $\partial N(\mathbb{L} \cup B)$ corresponding to the basepoints. A choice of framing of the link \mathbb{L} determines a diffeomorphism ϕ_λ of $\partial N(\mathbb{U} \cup B_0)$ to $\partial N(\mathbb{L} \cup B)$, which is compatible with ϕ_0 in the obvious sense. As we described after Definition 4.1, we can find parametrized surgery data for the tuple $(S^3 \setminus N(\mathbb{U} \cup B_0), Y \setminus (\mathbb{L} \cup B), \phi_\lambda)$. This consists of a framed 1-dimensional link $\mathbb{S}_1 \subseteq S^3 \setminus N(\mathbb{U} \cup B_0)$ and a diffeomorphism $f : (S^3 \setminus N(\mathbb{U} \cup B_0)) \rightarrow (Y \setminus N(\mathbb{L} \cup B))$ which extends ϕ_λ . This naturally induces a parametrized Kirby diagram $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ for \mathbb{L} , and a parametrized Kirby diagram $\mathbb{P}(B) = (\phi_0(B), \lambda(B), \mathbb{S}_1, f)$ for $(Y, \mathbb{L}(B))$, where $\phi_0(B)$ and $\lambda(B)$ are the diffeomorphisms and framings induced by ϕ_0, λ and B .

Assume now, for the sake of demonstration that B is a α -band for \mathbb{L} . We now pick a Heegaard quadruple $(\Sigma, \alpha', \alpha, \beta, \gamma)$ such that

- $(\Sigma, \alpha, \beta, \gamma)$ is a triple subordinate to a β -bouquet of the framed link $\mathbb{S}_1 \subseteq S^3 \setminus (\mathbb{U} \cup B)$;
- $(\Sigma, \alpha', \alpha, \beta)$ is a triple subordinate to the band B_0 for \mathbb{U} ;
- (the pushforward under f of) $(\Sigma, \alpha', \alpha, \gamma)$ is a triple subordinate to the band B for \mathbb{L} ;
- $(\Sigma, \alpha', \beta, \gamma)$ is a triple subordinate to a β -bouquet of the link $\mathbb{S}_1 \subseteq S^3 \setminus (\mathbb{U} \cup B)$.

The band B_0 connects two different components of \mathbb{U} , and $\mathbb{U}(B_0)$ is an unlink as well.

Let $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ be an intersection point. We wish to compute the grading of $F_{B^\circ}(\mathbf{x})$. Pick triangles

$$\begin{aligned} \psi_{\alpha'\alpha\beta} &\in \pi_2(\theta_{\alpha'\alpha}, \mathbf{x}_{\alpha\beta}, \mathbf{x}_{\alpha'\beta}), & \psi_{\alpha'\beta\gamma} &\in \pi_2(\mathbf{x}_{\alpha'\beta}, \theta_{\beta\gamma}, \mathbf{x}_{\alpha'\gamma}), & \psi_{\alpha\beta\gamma} &\in \pi_2(\mathbf{x}_{\alpha\beta}, \theta_{\beta\gamma}, \mathbf{x}), \\ \text{and} & & \psi_{\alpha'\alpha\gamma} &\in \pi_2(\theta_{\alpha'\alpha}, \mathbf{x}_{\alpha\gamma}, \mathbf{x}_{\alpha'\gamma}) \end{aligned}$$

with

$$\psi_{\alpha'\alpha\gamma} + \psi_{\alpha\beta\gamma} = \psi_{\alpha'\beta\gamma} + \psi_{\alpha'\alpha\beta}.$$

By splicing in disks to the ends of the triangles if necessary, we can assume that $\theta_{\alpha'\alpha}$ is in the same Alexander grading as Θ° and also that $\theta_{\beta\gamma}$ is in the same Alexander grading as the generator of top degree of $\mathcal{HFL}^-(\Sigma, \beta, \gamma)$. In fact, since these triangles represent the same Spin^c structures as those counted by F_B° and $F_{B_0}^\circ$, by splicing in disks on the ends, we can obtain any homology triangles which could be counted by the band maps F_B° or $F_{B_0}^\circ$.

Let $\widehat{\Sigma}$ denote the surface in $W(S^3, \mathbb{S}_1)$ obtained by capping off $\mathbb{U} \times [0, 1]$ with $f^{-1}(S)$ in $S^3(\mathbb{S}_1)$ and a J -Seifert surface in S^3 . Let $\widehat{\Sigma}'$ denote the surface in $W(S^3, \mathbb{S}_1)$ obtained by capping off $\mathbb{U}(B) \times [0, 1]$ with

$f^{-1}(S \cup B)$ and a J -Seifert surface in S^3 . Note that $\widehat{\Sigma}$ and $\widehat{\Sigma}'$ are homologous surfaces. We similarly define $\widehat{\Sigma}_j$ and $\widehat{\Sigma}'_j$ for each $j \in J$ and note that each $\widehat{\Sigma}_j$ is homologous to $\widehat{\Sigma}'_j$.

By definition of the absolute grading, we have that

$$A_{Y, \mathbb{L}(B), S \cup B}(\mathbf{x}_{\alpha'\gamma})_j = A(\mathbf{x}_{\alpha'\beta})_j + A(\theta_{\beta\gamma})_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi_{\alpha'\beta\gamma})_j + \frac{\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_{\alpha'\beta\gamma})), [\widehat{\Sigma}'_j] \rangle - [\widehat{\Sigma}'] \cdot [\widehat{\Sigma}'_j]}{2},$$

and

$$A_{Y, \mathbb{L}, S}(\mathbf{x})_j = A(\mathbf{x}_{\alpha\beta})_j + A(\theta_{\beta\gamma})_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi_{\alpha\beta\gamma})_j + \frac{\langle c_1(\mathfrak{s}_{\mathbf{w}}(\psi_{\alpha\beta\gamma})), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2}.$$

As we've remarked, the homological terms appearing at the end are equal. Hence we have just that

$$A_{Y, \mathbb{L}(B), S \cup B}(\mathbf{x}_{\alpha'\gamma})_j - A_{Y, \mathbb{L}, S}(\mathbf{x})_j = A(\mathbf{x}_{\alpha'\beta})_j - A(\mathbf{x}_{\alpha\beta})_j + (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi_{\alpha'\alpha\gamma} - \psi_{\alpha'\alpha\beta})_j,$$

which we can rewrite as

$$A_{Y, \mathbb{L}(B), S \cup B}(F_B^{\mathbf{o}}(\mathbf{x}))_j - A_{Y, \mathbb{L}, S}(\mathbf{x})_j = A(\mathbf{x}_{\alpha'\beta})_j - A(\mathbf{x}_{\alpha\beta})_j - (n_{\mathbf{w}} - n_{\mathbf{z}})(\psi_{\alpha'\alpha\beta})_j.$$

The above equation can be interpreted as saying that the grading change for the band maps $F_B^{\mathbf{o}}$ for $B \subseteq Y$ are the same as the grading change for the model band maps for the band B_0 , the band in S^3 between two components of an unlink, which is standard in the way we've described above.

This can now be performed as a model computation. As usual, we can pick any diagram to show this. The diagram we pick is obtained as follows. Suppose for moment that B_0 joins two components of \mathbb{U} . Pick a diagram for each component of \mathbb{U} in S^3 , then form a diagram for \mathbb{U} by attaching the diagrams together using 1-handles. If the ends of B_0 are adjacent to the basepoints w_1, w_2, z_1 and z_2 , then we can assume that we attach the 1-handles near z_1 and z_2 . A triple $(\Sigma, \alpha', \alpha, \beta)$ for the band B_0 is shown on the top of Figure 9.2, along with intersection points of $\Theta^{\mathbf{w}}$ and $\Theta^{\mathbf{z}}$. If B_0 splits a single component of \mathbb{U} , then we modify this procedure to get the Heegaard triple shown on the bottom row of Figure 9.2.

Outside of the region shown in Figure 9.2, curves α' are just small Hamiltonian isotopies of the curves α . In the top left of Figure 9.2, we show a portion of a triangle in $\pi_2(\Theta^{\mathbf{w}}, \Theta^{\mathbf{w}}, \Theta^{\mathbf{w}})$, which would be counted by $F_{B_0}^{\mathbf{z}}$. Outside of the region shown, we simply add in the small triangles (as the α' -curves are small isotopies of the α -curves). As this triangle class goes over no basepoints, we see that $F_{B_0}^{\mathbf{z}}$ maps the Alexander grading containing $\Theta^{\mathbf{w}}$ in $\mathcal{CFL}^-(\Sigma, \alpha, \beta)$ to the Alexander grading containing $\Theta^{\mathbf{w}}$ in $\mathcal{CFL}^-(\Sigma, \alpha', \beta)$. By our declaration of the absolute grading of unlinks in S^3 , this induces a grading change of $+\frac{1}{2}$ for the grading j_0 , and 0 in the other gradings.

A similar consideration of the other three triples shown in Figure 9.2 yields the rest of the grading changes. \square

Analogous to the previous lemma, we have the following:

Lemma 9.4. *Suppose (Y, \mathbb{L}) is a 3-manifold with multi-based link and J -Seifert surface S , and B is a band for \mathbb{L} , between link components with the same grading assignment, j_0 , and suppose that \mathfrak{s} is torsion. Then*

$$\text{gr}_{\mathbf{w}}(F_B^{\mathbf{z}}(\mathbf{x})) - \text{gr}_{\mathbf{w}}(\mathbf{x}) = 0, \quad \text{and} \quad \text{gr}_{\mathbf{w}}(F_B^{\mathbf{w}}(\mathbf{x})) - \text{gr}_{\mathbf{w}}(\mathbf{x}) = -1.$$

Similarly, if $\mathfrak{s} - PD[L]$ is torsion, then

$$\text{gr}_{\mathbf{z}}(F_B^{\mathbf{z}}(\mathbf{x})) - \text{gr}_{\mathbf{z}}(\mathbf{x}) = -1, \quad \text{and} \quad \text{gr}_{\mathbf{z}}(F_B^{\mathbf{w}}(\mathbf{x})) - \text{gr}_{\mathbf{z}}(\mathbf{x}) = 0.$$

Proof. Let us first consider $\text{gr}_{\mathbf{w}}$. The same argument as in the previous lemma reduces the model computation to computing the band map in S^3 between two unknots. The same model computation in Figure 9.2, together with our declaration of the absolute Maslov gradings for unlinks in S^3 immediately yields the stated grading change formulas. \square

Lemma 9.5. *The 0-handle and 4-handle maps $F_{0, \mathbb{U}}$ and $F_{4, \mathbb{U}}$, which add or remove a copy of (S^3, U, w, z) , are both 0-graded, for all gradings.*

Proof. One simply takes a parametrized Kirby diagram, and adds a copy of (S^3, \mathbb{U}) together with the empty framed link in S^3 , to the parametrized Kirby diagram. The formula for the grading is clearly unchanged. \square

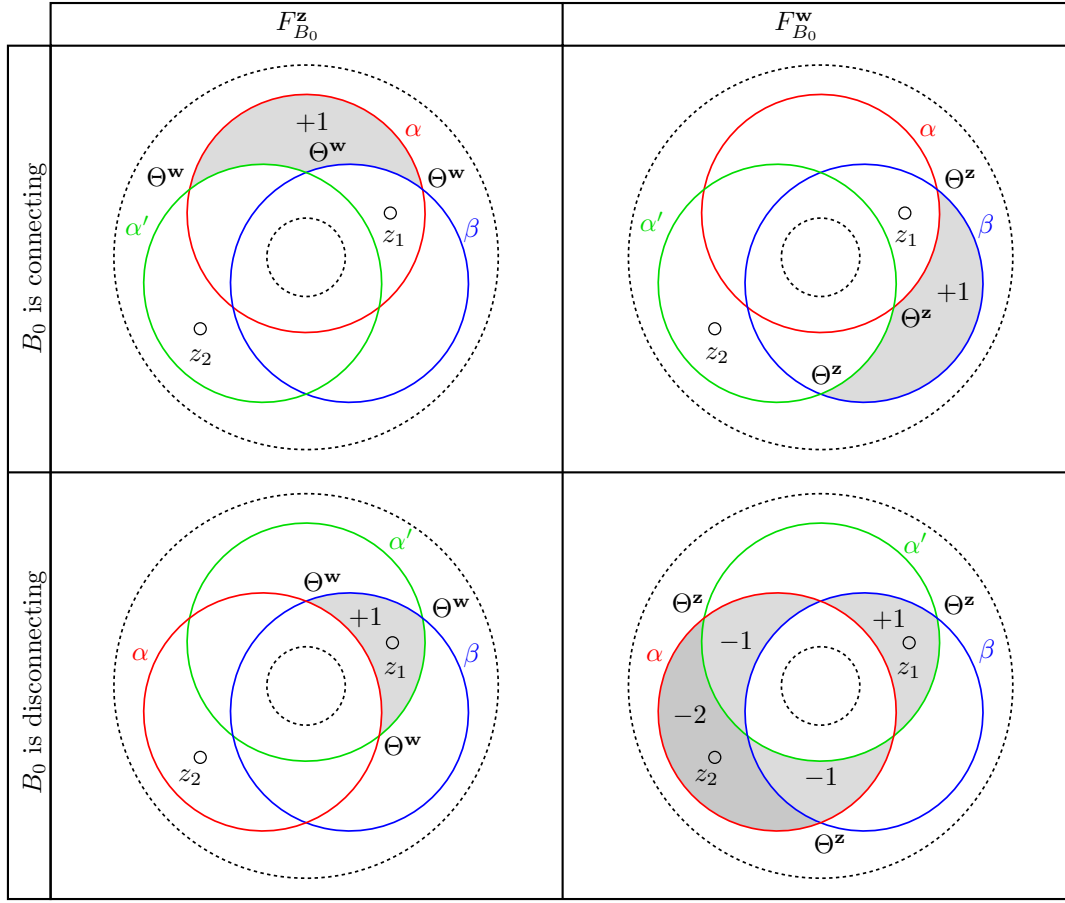


FIGURE 9.2. The triples used to compute the band maps $F_{B_0}^o$. Also shaded are portions of homology triangles. Outside of the 1-handle regions shown, the rest of the homology triangle consists of small triangles. The dashed circles are where the connected sum is taking place.

To package the grading changes induced by quasi-stabilization maps, band maps, and disk-stabilization maps, we define several homological quantities associated to surfaces with divides.

Recall that if (Σ, \mathcal{A}) is a surface with divides, then $\Sigma \setminus \mathcal{A}$ is partitioned into two subsurfaces, $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$. If $\pi : C(\Sigma) \rightarrow J$ is a grading assignment, let $\Sigma_{\mathbf{w},j}$ and $\Sigma_{\mathbf{z},j}$ denote the components of $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$ assigned the grading j . Given a link cobordism (W, F) from (Y_1, \mathbb{L}_1) to (Y_2, \mathbb{L}_2) with $\mathbb{L}_i = (L_i, \mathbf{w}_i, \mathbf{z}_i)$, we define the **reduced Euler characteristics**

$$\tilde{\chi}(\Sigma_{\mathbf{w},j}) = \chi(\Sigma_{\mathbf{w},j}) - \frac{1}{2}(|\mathbf{w}_1| + |\mathbf{w}_2|), \quad \text{and} \quad \tilde{\chi}(\Sigma_{\mathbf{z},j}) = \chi(\Sigma_{\mathbf{z},j}) - \frac{1}{2}(|\mathbf{z}_1| + |\mathbf{z}_2|).$$

Note that

$$\tilde{\chi}(\Sigma_{\mathbf{w},j}) - \tilde{\chi}(\Sigma_{\mathbf{z},j}) = \chi(\Sigma_{\mathbf{w},j}) - \chi(\Sigma_{\mathbf{z},j}).$$

We prove the following (cf. [HMZ, Lemma 4.3]):

Lemma 9.6. *The reduced Euler characteristics $\tilde{\chi}$ defined above are additive under composition of link cobordisms.*

Proof. This follows from the inclusion–exclusion principle on the Euler characteristic of a union. \square

We can now prove our Alexander grading formula for a general cobordism:

Theorem 1.5. *If $(W, F) : (Y_1, \mathbb{L}_1) \rightarrow (Y_2, \mathbb{L}_2)$ is a J -graded link cobordism which is given a J -graded coloring, and S_1 and S_2 are two J -Seifert surfaces for \mathbb{L}_1 and \mathbb{L}_2 , respectively, then we have the following:*

$$A_{Y_2, \mathbb{L}_2, S_2}(F_{W, F, \mathfrak{s}}^\circ(\mathbf{x}))_j - A_{Y_1, \mathbb{L}_1, S_1}(\mathbf{x})_j = \frac{\langle c_1(\mathfrak{s}), \widehat{\Sigma}_j \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2} + \frac{\chi(\Sigma_{\mathbf{w}, j}) - \chi(\Sigma_{\mathbf{z}, j})}{2}.$$

Proof. It is sufficient to verify that the formula is additive, and that it holds for the elementary cobordisms we've described. The homological terms involving $\widehat{\Sigma}_j$ are obviously additive. The terms involving the Euler characteristics are also easily seen to be additive, since each link component has the same number of \mathbf{w} basepoints as \mathbf{z} basepoints.

It remains to verify the formula for elementary cobordisms. Consider first a cobordism which is of the form $(Y \times [0, 1], L \times [0, 1])$ with a dividing set which has a single local max or local min along one of the arcs. The dividing sets are shown in Figure 9.1. The cobordism map is just a quasi-stabilization map, either $S_{w, z}^+, S_{w, z}^-, T_{w, z}^+$ or $T_{w, z}^-$, depending on the configuration of the arc with respect to the incoming or outgoing boundaries. For example if the arc has two endpoints on the outgoing boundary, and the small region bounded is of type $-\mathbf{w}$, then term $\frac{1}{2}(\chi(\Sigma_{\mathbf{w}, j}) - \chi(\Sigma_{\mathbf{z}, j}))$ is equal to $+\frac{1}{2}$. Similarly the grading change of the quasi-stabilization map corresponding to this dividing set, $S_{w, z}^+$, is also $+\frac{1}{2}$, according to Lemma 9.2. The other three configurations and quasi-stabilization maps can be analyzed similarly.

If the cobordism is diffeomorphic to $(Y \times [0, 2], (L \times [0, 1]) \cup B) \cup (Y \times [1, 2], L(B) \times [1, 2])$ (after smoothing the corners of the surface), with a dividing set of the form $\mathbf{p} \times [0, 1]$, and B is a type \mathbf{o} band (for $\mathbf{o} \in \{\mathbf{w}, \mathbf{z}\}$), then $\frac{1}{2}(\chi(\Sigma_{\mathbf{w}, j}) - \chi(\Sigma_{\mathbf{z}, j}))$ is equal to $+\frac{1}{2}$ if $\mathbf{o} = \mathbf{z}$, and is equal to $-\frac{1}{2}$ if $\mathbf{o} = \mathbf{w}$. This agrees with the grading of the band maps, as computed in Lemma 9.3.

Now suppose the cobordism is obtained by adding a 4-dimensional 1-handle away from the link, with both feet in the same component. We use the same Seifert surfaces on both sides. We stabilize the Heegaard triple by attaching a torus representing zero surgery on an unknot in a ball not intersecting the framed surgery link or the unlink U . An easy computation shows that the cobordism map induces Alexander grading change 0 if we use the same Seifert surface on both ends, agreeing with our formula for the grading. For a 1-handle with its feet attached in different components, we can just use the Heegaard triple formed by attaching the two components of the Heegaard surface with a 1-handle, and putting isotopic α_0, β_0 and γ_0 curves. A simple model computation shows that again the Alexander grading change is zero.

Similarly, if S is a Seifert surface which doesn't intersect a framed 2-sphere, then the 3-handle cobordism map is zero graded, with respect to the gradings induced by putting the same Seifert surface on both ends.

Finally if the cobordism is obtained by attaching 2-handles, then a homology associativity argument analogous to Lemma 6.1 shows that the change of grading in the j -component of the grading is

$$\frac{\langle c_1(\mathfrak{s}), \widehat{\Sigma}_j \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2}.$$

We finally consider 4-handle and 0-handle maps. By Lemma 9.5, the cobordism maps are 0-graded. On the other hand the terms involving the homology class $[\widehat{\Sigma}_j]$ vanish, and $\chi(\Sigma_{\mathbf{w}, j}) = \chi(\Sigma_{\mathbf{z}, j})$, so formula for the grading change reads zero as well.

Noting that (after removing some balls and applying the 0-handle or 4-handle maps) a parametrized Kirby decomposition in the sense of [Zem16b] gives us a decomposition in terms of the cobordisms described in the previous paragraph. The formula follows. \square

We can now prove our formula for the grading change of the Maslov gradings.

Theorem 1.6. *If $(W, F) : (Y_1, \mathbb{L}_1) \rightarrow (Y_2, \mathbb{L}_2)$ is a link cobordism with grading assignment and a J -graded coloring, and $\mathfrak{s} \in \text{Spin}^c(W)$ is a Spin^c structure which is torsion on Y_1 and Y_2 , then the link cobordism maps satisfy*

$$\text{gr}_{\mathbf{w}}(F_{W, F, \mathfrak{s}}^\circ(\mathbf{x})) - \text{gr}_{\mathbf{w}}(\mathbf{x}) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4} + \widetilde{\chi}(\Sigma_{\mathbf{w}}),$$

for homogeneous \mathbf{x} with respect to the $\text{gr}_{\mathbf{w}}$ grading. Similarly if $\mathfrak{s} - PD[\Sigma]$ restricts to a torsion Spin^c structure on the ends, then for \mathbf{x} which is homogeneous with respect to $\text{gr}_{\mathbf{z}}$, we have

$$\text{gr}_{\mathbf{z}}(F_{W, F, \mathfrak{s}}^\circ(\mathbf{x})) - \text{gr}_{\mathbf{z}}(\mathbf{x}) = \frac{c_1(\mathfrak{s} - PD[\Sigma])^2 - 2\chi(W) - 3\sigma(W)}{4} + \widetilde{\chi}(\Sigma_{\mathbf{z}}).$$

Proof. Let us first consider the $\text{gr}_{\mathbf{w}}$ grading. An easy computation shows that the decorated surfaces for associated to the quasi-stabilization maps $S_{w,z}^{\pm}$ satisfy $\tilde{\chi}(\Sigma_{\mathbf{w}}) = \frac{1}{2}$. Analogously, the maps $S_{w,z}^+$ and $S_{w,z}^-$ are $+\frac{1}{2}$ graded with respect to $\text{gr}_{\mathbf{w}}$, by Lemma 9.2. Similarly the dividing sets associated to the quasi-stabilization maps $T_{w,z}^+$ and $T_{w,z}^-$ have $\tilde{\chi}(\Sigma_{\mathbf{w}}) = -\frac{1}{2}$, while the maps $T_{w,z}^+$ and $T_{w,z}^-$ induce $\text{gr}_{\mathbf{w}}$ -grading change equal to $-\frac{1}{2}$, again by Lemma 9.2.

The band maps induce the expected grading change by Lemma 9.4. That 1- and 3-handles produce the expected grading change by adding a handle with three isotopic curves to the Heegaard surface involved in a surgery triple, as in [OS06]. The 2-handle maps count triangles with $\mathfrak{s}_{\mathbf{w}}(\psi) = \mathfrak{s}$, and hence induce the correct $\text{gr}_{\mathbf{w}}$ grading change, by the same argument as in [OS06]. The 0-handle and 4-handle maps are zero graded, while $\chi(W) = 1$ and $\tilde{\chi}(\Sigma_{\mathbf{w}}) = \frac{1}{2}$ for a 0-handle or 4-handle cobordism, so the proposed formula also reads zero. The formula for the grading change is additive under composition by Lemma 9.6, and hence the grading change formula for $\text{gr}_{\mathbf{w}}$ follows in general.

The same argument goes through for the $\text{gr}_{\mathbf{z}}$ grading, except in the case of 2-handles. The map associated to two handles counts triangles with $\mathfrak{s}_{\mathbf{w}}(\psi) = \mathfrak{s}$, whereas the absolute grading $\text{gr}_{\mathbf{z}}$ is defined using the Spin^c structure $\mathfrak{s}_{\mathbf{z}}(\psi)$. If ψ is counted by the 2-handle map, then $\mathfrak{s}_{\mathbf{w}}(\psi) = \mathfrak{s}$, while the 2-handle map induces $\text{gr}_{\mathbf{z}}$ grading change

$$\frac{c_1(\mathfrak{s}_{\mathbf{z}}(\psi))^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

Using Lemma 3.3, we see that $\mathfrak{s}_{\mathbf{z}}(\psi) = \mathfrak{s}_{\mathbf{w}}(\psi) - PD[\Sigma] = \mathfrak{s} - PD[\Sigma]$, and hence the $\text{gr}_{\mathbf{z}}$ grading will be changed by

$$\frac{c_1(\mathfrak{s} - PD[\Sigma])^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

Using additivity under composition of cobordisms, the proof is complete. \square

10. CONJUGATION SYMMETRY, EQUIVALENCE WITH OTHER CONSTRUCTIONS, AND COLLAPSING GRADINGS

In this section we describe several basic properties of the Alexander gradings we've defined, such as conjugation symmetry and the effect of collapsing the index set J .

10.1. Conjugation symmetry. We now prove that our construction of the absolute Alexander multi-grading agrees with the construction from definition in [OS08] by showing that it satisfies the conjugation symmetry property. Undoubtedly one could show this directly, but it's useful to have the conjugation symmetry result we state in this paper, so we state the equivalence of our grading and theirs as a consequence of this. In [OS08], the Alexander grading for an oriented link L in S^3 (with two basepoints per component) is presented over an affine \mathbb{Z}^ℓ lattice denoted $\mathbb{H}(L)$. Using the notation and Alexander gradings from [OS08, pg. 616], one has

$$(15) \quad \widehat{HFL}_d(L, h) = \widehat{HFL}_{d-2\delta(h)}(L, -h).$$

Here d denotes the homological grading, and h denotes the Alexander grading. The element h is an element of an affine, integral lattice $\mathbb{H}(L)$ and if $h = \sum_{i=1}^n a_i \cdot PD[\mu_i]$, (where μ_i is the meridian of the i^{th} link component and $a_i \in \mathbb{Q}$) then $\delta(h) = a_1 + \cdots + a_n$ (see [OS08] for more about this notation).

If $\mathbb{L} = (L, \mathbf{p}, \mathbf{q})$ is a multi-based link, then we let $\overline{\mathbb{L}} = (L, \mathbf{q}, \mathbf{p})$ denote the same link, but with the roles of the \mathbf{w} and \mathbf{z} basepoints switched. For notational simplicity, we will write U_t for the variable t associated to $t \in \mathbf{p} \cup \mathbf{q}$ and $\mathbb{Z}_2[U_{\mathbf{p} \cup \mathbf{q}}]$ for the associated ring. There is a natural conjugation action on $\text{Spin}^c(Y)$. To a non-vanishing vector field v representing \mathfrak{s} , we define $\bar{\mathfrak{s}}$ to be the Spin^c structure determined by $-v$.

We now describe the conjugation action η on \mathcal{CFL}° . Given a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{p}, \mathbf{q})$ for $\mathbb{L} = (L, \mathbf{p}, \mathbf{q})$, we consider the conjugate diagram $\overline{\mathcal{H}} = (-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{q}, \mathbf{p})$ for $(Y, \overline{\mathbb{L}})$. There is an obvious correspondence between the intersection points on the two diagrams. We note however, that $\overline{\mathfrak{s}_{\mathcal{H}, \mathbf{p}}(\mathbf{x})} = \mathfrak{s}_{\overline{\mathcal{H}}, \mathbf{p}}(\eta(\mathbf{x}))$, by explicit examination of the vector fields constructed by Ozsváth and Szabó in [OS04b], but the types (\mathbf{w} or \mathbf{z}) of the basepoints \mathbf{p} and \mathbf{q} has been switched. On the other hand,

$$\mathfrak{s}_{\overline{\mathcal{H}}, \mathbf{q}}(\eta(\mathbf{x})) = \overline{\mathfrak{s}_{\mathcal{H}, \mathbf{p}}(\mathbf{x})} + PD[L],$$

from Lemma 3.2. We extend the map η to arbitrary generators (not just intersection points) by declaring it to be linear in the U_t variables for $t \in \mathbf{p} \cup \mathbf{q}$. By the above formula of Spin^c structures, we see that η maps

$\mathcal{CFL}^\circ(Y, \mathbb{L}, \mathfrak{s})$ to $\mathcal{CFL}^\circ(Y, \overline{\mathbb{L}}, \bar{\mathfrak{s}} + PD[L])$. As in [HM15], the map η commutes with change of diagrams maps, up to equivariant filtered chain homotopy.

In this section, we prove the following:

Proposition 10.1. *Suppose that $\mathbb{L} = (L, \mathbf{p}, \mathbf{q})$ is a link in Y with a grading assignment $\pi : C(L) \rightarrow J$, which is J -null-homologous. The map η induces a chain homotopy equivalence $\eta : \mathcal{CFL}^\circ(Y, \mathbb{L}, \mathfrak{s}) \rightarrow \mathcal{CFL}^\circ(Y, \overline{\mathbb{L}}, \bar{\mathfrak{s}})$ which is $\mathbb{Z}_2[U_{\mathbf{p} \cup \mathbf{q}}]$ -equivariant and $\mathbb{Z}^{\mathbf{p}} \oplus \mathbb{Z}^{\mathbf{q}}$ filtered. The map η maps homogeneous graded elements (with respect to $A_{Y, \mathbb{L}, S}$, $\text{gr}_{\mathbf{w}}$ or $\text{gr}_{\mathbf{z}}$) to homogeneous elements, and satisfies*

$$A_{Y, \overline{\mathbb{L}}, S}(\eta(\mathbf{x})) = -A_{Y, \mathbb{L}, S}(\mathbf{x}), \quad \text{gr}_{\mathbf{w}}(\eta(\mathbf{x})) = \text{gr}_{\mathbf{z}}(\mathbf{x}), \quad \text{and} \quad \text{gr}_{\mathbf{z}}(\eta(\mathbf{x})) = \text{gr}_{\mathbf{w}}(\mathbf{x}),$$

for a J -Seifert surface S .

Proof. We first consider how η affects the relative Alexander and Maslov gradings. We note that since \mathbb{L} and $\overline{\mathbb{L}}$ have the roles of the type- \mathbf{w} basepoints and type- \mathbf{z} basepoints switched, an easy computation shows that the relative gradings satisfy

$$A(\eta(x), \eta(y))_j = -A(x, y)_j, \quad \text{gr}_{\mathbf{w}}(\eta(x), \eta(y)) = \text{gr}_{\mathbf{z}}(x, y), \quad \text{and} \quad \text{gr}_{\mathbf{z}}(\eta(x), \eta(y)) = \text{gr}_{\mathbf{w}}(x, y),$$

whenever x and y are homogeneous with respect to the appropriate gradings. Thus it is sufficient to just consider how η affects the absolute gradings of intersection points.

We take a parametrized Kirby diagram $\mathbb{P} = (\phi_0, \lambda, \mathbb{S}_1, f)$ and a surgery triple $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{p}, \mathbf{q})$, which is subordinate to a β -bouquet for \mathbb{S}_1 . If $\mathbf{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\gamma}$, and $\psi \in \pi_2(\mathbf{x}, \theta, \mathbf{y})$ is a homology triangle, with any other intersection points \mathbf{x} and θ , then the absolute Alexander gradings for $\mathcal{CFL}^\circ(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{w}, \mathbf{z})$ are defined by the formula

$$A_{Y, \mathbb{L}, S}(\mathbf{y})_j = A(\mathbf{x})_j + A(\theta)_j + (n_{\mathbf{p}}(\psi) - n_{\mathbf{q}}(\psi))_j + \frac{\langle c_1(\mathfrak{s}_{\mathbf{p}}(\psi)), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2}.$$

We can form the conjugate triple $\overline{\mathcal{T}} = (-\Sigma, \overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\alpha}}, \mathbf{q}, \mathbf{p})$ of \mathcal{T} . Here the $\overline{\boldsymbol{\alpha}}$ curves are the same on $-\Sigma$ as the $\boldsymbol{\alpha}$ curves on Σ , and similarly for the other sets of curves, but the roles of the various curves is now changed. The β -bouquet for \mathbb{S}_1 now becomes an α -bouquet for \mathbb{S}_1 , and $\overline{\mathcal{T}}$ is now subordinate to this α -bouquet for the same framed link \mathbb{S}_1 in $S^3 \setminus U$.

Using Lemma 7.11, we can use $\overline{\mathcal{T}}$ to compute the grading of $\eta(\mathbf{y})$. Notice that there is a canonical identification

$$X_{\alpha\beta\gamma} \cong X_{\overline{\gamma}\overline{\beta}\overline{\alpha}},$$

as oriented manifolds, and similarly this identification respects the embedding of both in $W(S^3, \mathbb{S}_1)$. This diffeomorphism also identifies

$$\Sigma_{\alpha\beta\gamma} \cong \Sigma_{\overline{\gamma}\overline{\beta}\overline{\alpha}},$$

as oriented surfaces, and their images inside of $W(S^3, \mathbb{S}_1)$ coincide as subsurfaces of $\widehat{\Sigma}$. The homology triangle ψ induces a triangle, $\overline{\psi}$, on the conjugate Heegaard triple.

The Alexander grading of $\eta(\mathbf{y})$ can be computed using $\overline{\mathcal{T}}$ and the triangle $\overline{\psi}$, and indeed we see that

$$A(\eta(\mathbf{y}))_j = A(\eta(\mathbf{x}))_j + A(\eta(\theta))_j + (n_{\mathbf{q}}(\overline{\psi}) - n_{\mathbf{p}}(\overline{\psi}))_j + \frac{\langle c_1(\mathfrak{s}_{\mathbf{q}}(\overline{\psi})), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]}{2}.$$

We claim that $A(\eta(\mathbf{y}))_j = -A(\mathbf{y})_j$, from which the claim will follow. First observe that $A(\eta(\mathbf{x})) = -A(\mathbf{x})$ and similarly for θ , by an easy model computation for unlinks in S^3 or $(S^1 \times S^2)^{\#k}$ (the map η and the Alexander gradings are natural, so such a computation can be checked on any convenient diagram).

Note that obviously

$$(n_{\mathbf{q}}(\psi) - n_{\mathbf{p}}(\psi))_j = -(n_{\mathbf{p}}(\overline{\psi}) - n_{\mathbf{q}}(\overline{\psi}))_j.$$

We now consider the homological terms involving the homology class of the surface $\Sigma_{\alpha\beta\gamma}$ appearing in the formula for the grading. We note that $\mathfrak{s}_{\mathbf{p}}(\overline{\psi}) = \overline{\mathfrak{s}_{\mathbf{p}}(\psi)}$, the conjugate Spin^c structure. Similarly

$\mathfrak{s}_{\mathbf{q}}(\bar{\psi}) = \mathfrak{s}_{\mathbf{p}}(\bar{\psi}) + PD[\Sigma_{\bar{\gamma}\bar{\beta}\bar{\alpha}}]$ by Lemma 3.3. Note that $\Sigma_{\alpha\beta\gamma}$ can be capped off with J -Seifert surfaces in the ends of $X_{\alpha\beta\gamma}$ to yield a surface which is homologous to $\widehat{\Sigma}$. Hence we just compute

$$\begin{aligned} \langle c_1(\mathfrak{s}_{\mathbf{q}}(\bar{\psi})), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j] &= \langle c_1(\mathfrak{s}_{\mathbf{p}}(\bar{\psi}) + PD[\widehat{\Sigma}]), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j] \\ &= \langle c_1(\mathfrak{s}_{\mathbf{p}}(\bar{\psi}) + PD[\widehat{\Sigma}]), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j] \\ &= -\langle c_1(\mathfrak{s}_{\mathbf{p}}(\psi)), [\widehat{\Sigma}_j] \rangle + 2\langle PD[\widehat{\Sigma}], [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j] \\ &= -(\langle c_1(\mathfrak{s}_{\mathbf{p}}(\psi)), [\widehat{\Sigma}_j] \rangle - [\widehat{\Sigma}] \cdot [\widehat{\Sigma}_j]). \end{aligned}$$

Thus each term in $A(\mathbf{y})_j$ is changed to its negative in $A(\eta(\mathbf{y}))_j$, from which we conclude that $A(\eta(\mathbf{y}))_j = -A(\mathbf{y})_j$.

The proof of the statement about the Maslov gradings $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ follows similarly. \square

The map η restricts to the \mathbb{Z}_2 -modules \widehat{CFL} to thus yield an isomorphism between $\widehat{HFL}(Y, \mathbb{L})_s$ and $\widehat{HFL}(Y, \mathbb{L})_{-s}$ where s denotes our Alexander grading. Since the gradings from [OS08] also satisfy this relation with respect to the conjugation action, the two gradings must coincide.

Indeed we also note that the homological grading shift from Equation (15) can be read off in our setting as well, as $\text{gr}_{\mathbf{w}}(\eta(x)) = \text{gr}_{\mathbf{z}}(x) = \text{gr}_{\mathbf{w}}(x) - 2A(x)$, using our notation.

10.2. Collapsing gradings. Finally we make a remark on collapsing grading assignments. If $\pi : L \rightarrow J$ is a grading assignment, and $f : J \rightarrow J'$ is a map, then the map $\pi' = f \circ \pi : L \rightarrow J'$ is also a grading assignment.

Lemma 10.2. *The multi-grading $A_{Y, \mathbb{L}, S, \pi'}$ is obtained from $A_{Y, \mathbb{L}, S, \pi}$ by collapsing gradings, in the sense that*

$$(A_{Y, \mathbb{L}, S, \pi'})_{j'} = \sum_{j \in f^{-1}(j')} (A_{Y, \mathbb{L}, S, \pi})_j.$$

The empty sum is interpreted as zero.

Proof. This follows from the fact that the stated equation holds for unlinks in $(S^1 \times S^2)^{\#k}$ by Equation (8), and the fact that the expressions $\langle c_1(\mathfrak{s}), \widehat{\Sigma}_j \rangle$ and $[\Sigma] \cdot [\Sigma_j]$ each also satisfy the analogous equation associated to collapsing gradings. \square

11. A NEW PROOF OF A BOUND OF OZSVÁTH AND SZABÓ ON τ

In [OS03b], Ozsváth and Szabó define a homomorphism $\tau : \mathcal{C} \rightarrow \mathbb{Z}$, and prove a bound on how τ behaves with respect to knot cobordisms in negative definite 4-manifolds. In this section, we give a new proof of this bound, as a consequence of our grading formula.

We briefly recall the definition of τ and some other notation. The differential on $\widehat{CFL}(S^3, K, w, z)$ is defined by counting disks which don't go over any of the w or z basepoints. If we allow disks to go over z but not w , the resulting chain complex is $\widehat{CF}(S^3, w)$, but with a \mathbb{Z} -filtration. Given a diagram \mathcal{H} for (S^3, K) , we let $\mathcal{F}_s(\mathcal{H}, K)$, denote the subgroup of $\widehat{CF}(\mathcal{H})$ generated by \mathbf{x} with $A(\mathbf{x}) \leq s$. The filtered chain homotopy type is a knot invariant, so we will omit \mathcal{H} from the notation, and write $\mathcal{F}_s(K)$. The integer $\tau(K)$ is defined as the minimal s such that the map

$$H_*(\mathcal{F}_s(K)) \rightarrow \widehat{HF}(S^3) \cong \mathbb{Z}_2$$

induced by inclusion is nontrivial. It is easy to check that when we set $V = 1$, and $U = 0$ a map $F : CFL^-(S^3, K_1) \rightarrow CFL^-(S^3, K_2)$ which changes Alexander grading by $+\Delta$ induces a map $\mathcal{F}_s(K_1) \rightarrow \mathcal{F}_{s+\Delta}(K_2)$. If we can ensure the induced map $\widehat{HF}(S^3) \rightarrow \widehat{HF}(S^3)$ is an isomorphism, then we can quickly obtain bounds on the invariant τ .

Suppose as in [OS03b] that W is an oriented 4-manifold with $\partial W = S^3$ and $b_2^+(W) = b_1(W) = 0$. If e_1, \dots, e_b is an orthonormal basis of $H^2(W; \mathbb{Z})$, and

$$[\Sigma] = s_1 \cdot e_1 + \dots + s_b \cdot e_b,$$

we define

$$|[\Sigma]| = |s_1| + \cdots + |s_b|,$$

a well defined integer (see [OS03b]).

The following theorem is essentially [OS03b, Theorem 1.1], for oriented surfaces in negative definite 4-manifolds. We give a new proof, using our grading formula:

Theorem 11.1. *Suppose that $(W, \Sigma) : (S^3, K_1) \rightarrow (S^3, K_2)$ is an oriented knot cobordism with $b_2^+(W) = 0 = b_1(W)$. Then*

$$\tau(K_2) \leq \tau(K_1) - \frac{|[\Sigma]| + [\Sigma] \cdot [\Sigma]}{2} + g(\Sigma).$$

Proof. As above, we write $b = b_2(W) = b_2^-(W)$. Decorate Σ by picking a path from one boundary to the other, and letting $\Sigma_{\mathbf{w}}$ be a regular neighborhood of this path. Let F be the resulting decorated link cobordism. Using [OS03a, Section 9], if \mathfrak{s} is a Spin^c structure on $W \setminus B$ with $c_1(\mathfrak{s})^2 + b = 0$, then the map $\widehat{F}_{W, \gamma, \mathfrak{s}} : \widehat{HF}(S^3) \rightarrow \widehat{HF}(S^3)$ is an isomorphism. Here γ is the path we chose on Σ (though the fact that the map is an isomorphism doesn't depend on the path). By the grading formula, the maps $F_{W, F, \mathfrak{s}}$ induce a map from $\mathcal{F}_{\tau(K_1)}(K_1)$ to $\mathcal{F}_{\tau(K_1) + \Delta}(K_2)$ which is equal to the map $\widehat{F}_{W, \mathfrak{s}}$ on homology, where the grading change Δ is given by the formula

$$\Delta = \frac{\langle c_1(\mathfrak{s}), [\Sigma] \rangle - [\Sigma] \cdot [\Sigma]}{2} + g(\Sigma).$$

There are 2^b characteristic vectors C with $C \cdot C = -b$ (these are sums of ± 1 times each e_i), and upon direct inspection, we see that

$$-|[\Sigma]| = \min_{C \in \text{Char}(W): C \cdot C = -b} \langle C, [\Sigma] \rangle.$$

Also every characteristic vector of the intersection form is the first Chern class of a Spin^c structure. Pick an $\mathfrak{s} \in \text{Spin}^c(W)$ with $c_1(\mathfrak{s})^2 + b = 0$ such that $c_1(\mathfrak{s})$ realizes the above equality. Pick an $a \in \mathcal{F}_{\tau(K_1)}$ such that $[a] \neq 0 \in \widehat{HF}(S^3)$. Then $F_{W, F, \mathfrak{s}}(a) \in \mathcal{F}_{\tau(K_1) + \Delta}(K_2)$ is an element which is nonzero in $\widehat{HF}(S^3)$. Hence

$$\tau(K_2) \leq \tau(K_1) + \Delta = \tau(K_1) + \frac{-|[\Sigma]| - [\Sigma] \cdot [\Sigma]}{2} + g(\Sigma).$$

□

For example, if $(W, \Sigma) : (S^3, K_1) \rightarrow (S^3, K_2)$ is a knot cobordism with $g(\Sigma) = g$ and Σ connected, such that W is a rational homology cobordism, then $|\tau(K_1) - \tau(K_2)| \leq g$.

In light of the proof of the previous bound, we see that homomorphism $\tau : \mathcal{C} \rightarrow \mathbb{Z}$ naturally factors through the **integer homology concordance group** $\mathcal{C}_{\mathbb{Z}}^3$, which is generated by pairs (Y^3, K) of knots in integer homology spheres modulo the relation that two pairs are equivalent if there is an integer homology cobordism between the 3-manifolds, which contains an oriented genus zero surface between the two knots.

12. t -MODIFIED KNOT FLOER HOMOLOGY AND A BOUND ON THE $\Upsilon_K(t)$ INVARIANT

In [OSS14], Ozsváth, Stipsicz and Szabó define a homomorphism from the smooth concordance group \mathcal{C} to the group of piecewise linear, \mathbb{R} -valued functions over $[0, 2]$. In this section, we recall their construction, and show how our grading formula and link cobordism maps yields a bound for $\Upsilon_K(t)$.

We recall (a version of) their construction. Suppose that $K \subseteq S^3$ is an oriented knot, and $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ is a diagram for (S^3, K, w, z) . If $0 \leq t = \frac{m}{n} \leq 2$, we define the t -grading on intersection points $\mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$ by

$$\text{gr}_t(\mathbf{x}) = (1 - \frac{t}{2}) \text{gr}_{\mathbf{w}}(\mathbf{x}) + \frac{t}{2} \text{gr}_{\mathbf{z}}(\mathbf{x}).$$

Note that this agrees with the formula from [OSS14]. We then consider the chain complex $tCFK^-(K)$, a chain complex over $\mathbb{Z}_2[v^{1/n}]$, generated by intersection points $\mathbf{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$ with differential

$$\partial(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1}} \# \widehat{\mathcal{M}}(\phi) v^{tn_z(\phi) + (2-t)n_w(\phi)} \cdot \mathbf{y}.$$

The grading gr_t descends to give a grading on $tCFK^-(K)$. The differential lowers degree by 1, and the action of v also lowers degree by 1. The number $\Upsilon_K(t) \in \mathbb{R}$ is defined as the maximal gr_t -grading of any homogeneous non-torsion element of $tHFK^-(K)$.

To obtain bounds on $\Upsilon_K(t)$, we first show that we can recover $tCFK^-(K)$ from $CFL^-(S^3, K, w, z)$. Define the ring

$$R_t^- = \mathbb{Z}_2[U, V, v^{1/n}]/(U - v^{2-t}, V - v^t),$$

and define R_t^∞ similarly.

Lemma 12.1. *The chain complex $CFL^-(S^3, K, w, z) \otimes_{\mathbb{Z}_2[U, V]} R_t^-$ is naturally isomorphic to $tCFK^-(K)$ as a $\mathbb{Z}_2[v^{1/n}]$ -module.*

Proof. We first describe an isomorphism between the rings R_t^- and $\mathbb{Z}_2[v^{1/n}]$. Noting that $t = \frac{m}{n}$, the map from $\mathbb{Z}_2[U, V, v^{1/n}]/(U - v^{2-t}, V - v^t)$ to $\mathbb{Z}_2[v^{1/n}]$ is

$$U^i V^j v^s \mapsto v^{i(2-t)+jt+s}$$

and a map in the opposite direction is

$$v^s \mapsto v^s.$$

To define maps between the chain complexes, we use the above maps on rings, extended over linear combinations of intersection points. That these maps are chain maps is immediate. It's also clear that these two maps are inverses of each other. \square

Phrased another way, R_t^- is in fact just $\mathbb{Z}_2[v^{1/n}]$ with a module action of $\mathbb{Z}_2[U, V]$ declared. In a similar manner the chain complex $tCFK^\infty(S^3, K)$ is isomorphic to $CFL^\infty(S^3, K) \otimes_{\mathbb{Z}_2[U, V, U^{-1}, V^{-1}]} R_t^\infty$ as a $\mathbb{Z}_2[v^{1/n}, v^{-1/n}]$ -module.

As a consequence, if $(W, \Sigma) : (S^3, K_1) \rightarrow (S^3, K_2)$ is a link cobordism and $\mathfrak{s} \in \text{Spin}^c(W)$, then the link cobordism maps $F_{W, \mathfrak{s}}^-$ induce a map from $tCFK^-(K_1)$ to $tCFK^-(K_2)$.

Lemma 12.2. *Suppose that (W, Σ) is a knot cobordism from (S^3, K_1) to (S^3, K_2) and $\mathfrak{s} \in \text{Spin}^c(W)$. If the 3-manifold cobordism map $F_{W, \mathfrak{s}}^\infty$ is an isomorphism on $HFK^\infty(S^3)$, and we decorate Σ with a dividing set consisting of two arcs going from K_1 to K_2 , such that the induced region $\Sigma_{\mathbf{w}}$ is equal to a strip on Σ from K_1 to K_2 , then the link cobordism map*

$$F_{W, \mathfrak{s}}^- : tHFK^-(K_1) \rightarrow tHFK^-(K_2)$$

sends non-torsion elements of $tHFK^-(K_1)$ to non-torsion elements of $tHFK^-(K_2)$.

Proof. We consider $tCFK^\infty(K_i)$, which are modules over $\mathbb{Z}_2[v^{-1/n}, v^{1/n}]$. There is a natural map from $tHFK^-(K_i)$ to $tHFK^\infty(K_i)$, given by inclusion of complexes. It's easily seen that an element of $tHFK^-(K)$ is non-torsion iff it maps to a nonzero element of $tHFK^\infty(K)$. On the other hand, there is an isomorphism

$$tHFK^\infty(K) \cong H_*(CFL^\infty(S^3, K) \otimes_{\mathbb{Z}_2[U, V, U^{-1}, V^{-1}]} R_t^\infty).$$

We claim that this is isomorphic to just $H_*(\mathcal{HFL}^\infty(S^3, K)) \otimes_{\mathbb{Z}_2[U, V, U^{-1}, V^{-1}]} R_t^\infty$. To see this, it's easiest to note that $CFL^\infty(S^3, K)$ decomposes over Alexander gradings, and the 0-graded part is the standard $CFK^\infty(S^3, K)$ complex, viewed as a module over $\mathbb{Z}_2[\bar{U}, \bar{U}^{-1}]$ where $\bar{U} = UV$. Using the classification of chain complexes over the PID $\mathbb{Z}_2[\bar{U}, \bar{U}^{-1}]$ (see [HMZ, Lemma 6.1]) and extending over all of $\mathbb{Z}_2[U, V, U^{-1}, V^{-1}]$, we can write $CFL^\infty(S^3, K)$ as a sum of two step complexes $(a_i \rightarrow b_i)$ for $i = 1, \dots, n$, as well as a single summand (a_0) with vanishing differential in zero $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$ gradings (of course this decomposition doesn't respect the $\mathbb{Z} \oplus \mathbb{Z}$ filtration). Tensoring with R_t^∞ preserves this decomposition of the chain complex, and hence $tHFK^\infty(S^3, K)$ is the free $\mathbb{Z}_2[v^{1/n}, v^{-1/n}]$ -module generated by $[a_0]$.

Phrased another way, the natural map

$$\mathcal{HFL}^\infty(S^3, k) \otimes_{\mathbb{Z}_2[U, V, U^{-1}, V^{-1}]} R_t^\infty \rightarrow tHFK^\infty(K)$$

is an isomorphism

Additionally, since $\mathcal{HFL}^\infty(S^3, K_i) \cong \mathbb{Z}_2[U, V, U^{-1}, V^{-1}]$, on homology the link cobordism map is determined by its value on $1 \in \mathbb{Z}_2[U, V, U^{-1}, V^{-1}] \cong \mathcal{HFL}^\infty(S^3, K_1)$. Since the link cobordism maps are graded, $F_{W, \mathfrak{s}}^\infty$ must be $c \cdot [x]$, where $[x]$ is a nonzero element of $\mathcal{HFL}^\infty(S^3, K_2)$ in the expected grading and $c \in \mathbb{Z}_2$.

As $F_{W,F,\mathfrak{s}}^\infty$ covers $F_{W,\mathfrak{s}}^\infty$ when we set $V = 1$, we see that $c = 1$ since by assumption $F_{W,\mathfrak{s}}^\infty$ is an isomorphism. In particular, the map induced by $F_{W,F,\mathfrak{s}}^\infty$ on $tHFL^\infty$ is an isomorphism. From these considerations applied to the commutative diagram

$$\begin{array}{ccc} tHFK^-(K_1) & \xrightarrow{\iota} & tHFK^\infty(K_1) \\ \downarrow F_{W,F,\mathfrak{s}}^- & & \cong \downarrow F_{W,F,\mathfrak{s}}^\infty \\ tHFK^-(K_2) & \xrightarrow{\iota} & tHFK^\infty(K_2), \end{array}$$

the claim now follows. \square

Finally, we note that if (W, F, \mathfrak{s}) is a link cobordism, the gr_t -grading change can be computed using the $\text{gr}_{\mathbf{w}}$ - and $\text{gr}_{\mathbf{z}}$ -grading changes we've computed in previous sections, as gr_t is a convex combination of $\text{gr}_{\mathbf{x}}$ and $\text{gr}_{\mathbf{z}}$. We compute that if \mathbf{x} is a homogeneously graded element, then

$$(16) \quad \begin{aligned} & \text{gr}_t(F_{W,F,\mathfrak{s}}^-(\mathbf{x})) - \text{gr}_t(\mathbf{x}) \\ &= \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4} + t \cdot \left(\frac{-\langle c_1(\mathfrak{s}), \widehat{\Sigma} \rangle + [\widehat{\Sigma}] \cdot [\widehat{\Sigma}]}{2} \right) + \left(1 - \frac{t}{2} \right) \cdot \tilde{\chi}(\Sigma_{\mathbf{w}}) + \frac{t}{2} \cdot \tilde{\chi}(\Sigma_{\mathbf{z}}), \end{aligned}$$

using the grading change formulas for $\text{gr}_{\mathbf{w}}$ and $\text{gr}_{\mathbf{z}}$.

We now proceed to prove Theorem 1.1. Recall from the introduction that if s is an integer and $t \in [0, 2]$, we define

$$M_t(s) = \max_{a \in 2\mathbb{Z}+1} \frac{-a^2 + 1 + 2ast - 2s^2t}{4}.$$

If e_1, \dots, e_n is an orthonormal basis for $H_2(W; \mathbb{Z})$, where W is a negative definite 4-manifold, and $[\Sigma] = s_1 \cdot e_1 + \dots + s_n \cdot e_n \in H_2(W; \mathbb{Z})$, we define

$$M_t([\Sigma]) = \sum_{i=1}^n M_t(s_i).$$

Theorem 1.1. *Suppose that $(W, \Sigma) : (S^3, K_1) \rightarrow (S^3, K_2)$ is an oriented knot cobordism with $b_2^+(W) = 0 = b_1(W)$. Then*

$$\Upsilon_{K_2}(t) \geq \Upsilon_{K_1}(t) + M_t([\Sigma]) + g(\Sigma) \cdot (|t - 1| - 1).$$

Proof. We construct a decorated link cobordism (W, F) with $F = (\Sigma, \mathcal{A})$, where the divides \mathcal{A} are the boundary of a neighborhood of a path on the surface from K_1 to K_2 , which we let designate as type- \mathbf{w} . By the proof [OS03a, Theorem 9.1], the map $F_{W,\mathfrak{s}}^\infty : HF^\infty(S^3) \rightarrow HF^\infty(S^3)$ is an isomorphism for any Spin^c structure $\mathfrak{s} \in \text{Spin}^c(W)$. By Lemma 12.2, the map on $tHFL^-$ induced by the link cobordism map $F_{W,F,\mathfrak{s}}^-$ sends non-torsion elements to non-torsion elements, for each t . Using the gr_t -grading change from Equation (16), the map $F_{W,F,\mathfrak{s}}^-$ induces a gr_t -grading change of

$$\frac{c_1(\mathfrak{s})^2 + b_2(W) - 2t \cdot \langle c_1(\mathfrak{s}), [\Sigma] \rangle + 2t \cdot [\Sigma] \cdot [\Sigma]}{4} - t \cdot g(\Sigma)$$

on $tHFL^-$. The set of Chern classes of Spin^c structures on W is equal to the set of characteristic vectors. We note that the set of characteristic vectors in $H^2(W; \mathbb{Z})$ is equal to (the Poincaré duals of) the elements $a_1 \cdot e_1 + \dots + a_b \cdot e_b$ where each a_i is an odd integer. Thus if $PD[c_1(\mathfrak{s})] = a_1 \cdot e_1 + \dots + a_b \cdot e_b$, we have that

$$\frac{c_1(\mathfrak{s})^2 + b_2(W) - 2t \cdot \langle c_1(\mathfrak{s}), [\Sigma] \rangle + 2t \cdot [\Sigma] \cdot [\Sigma]}{4} = \sum_{i=1}^b \frac{-a_i^2 + 1 + 2ta_i s_i - 2ts_i^2}{4}.$$

Since the summands are independent from each other in i , we can maximize together or separately. Taking the maximum over $\mathfrak{s} \in \text{Spin}^c(W)$ for a fixed t , we get

$$\Upsilon_{K_2}(t) \geq \Upsilon_{K_1}(t) + M_t([\Sigma]) - t \cdot g(\Sigma).$$

Letting $F_a(t) = \frac{1}{4}(-a^2 + 1 + 2ast - 2s^2t)$, we note that

$$F_a(t) = F_{-a+2s}(2-t),$$

so that $M_t(n) = M_{2-t}(n)$. Using also the symmetry of the $\Upsilon_K(t)$ invariant, we see that

$$\Upsilon_{K_2}(t) \geq \Upsilon_{K_1}(t) + M_t([\Sigma]) - (2-t) \cdot g(\Sigma).$$

Combining this bound with first yields the stated inequality. \square

Remark 12.3. One could hope to refine the above bound even further, by considering different surfaces with divides and trying to optimize the expression

$$\left(1 - \frac{t}{2}\right) \cdot \tilde{\chi}(\Sigma_{\mathbf{w}}) + \frac{t}{2} \cdot \tilde{\chi}(\Sigma_{\mathbf{z}}).$$

We now remark that this doesn't yield any stronger bounds. If we pick a dividing set on Σ consisting of two arcs from K_1 to K_2 which cut it into two connected components $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$, then $\tilde{\chi}(\Sigma_{\mathbf{w}}) = -2g(\Sigma_{\mathbf{w}})$ and similarly for $\Sigma_{\mathbf{z}}$. Adapting Theorem 1.8, which we prove in the next section, we can actually see that the map $F_{W,F,s}^\infty$ on \mathcal{HFL}^∞ is an isomorphism. On the other hand, using $k = g(\Sigma_{\mathbf{w}})$, we see that we can replace the summand $-t \cdot g(\Sigma)$ appearing in the gr_t -grading change formula with

$$\max_{\substack{k \in \mathbb{Z} \\ 0 \leq k \leq g(\Sigma)}} -(2-t)k - t(g(\Sigma) - k) = g(\Sigma) \cdot (|t-1| - 1),$$

so the bound we get by varying the set of divides is no better than the one we proved using the divides where $\Sigma_{\mathbf{w}}$ is a strip on Σ , and then using the symmetry of $\Upsilon_K(t)$.

12.1. Additional examples of the bound. As a corollary, we see that the above theorem recovers the genus bound from [OSS14]:

Corollary 12.4. *If $(W, \Sigma) : (S^3, K_1) \rightarrow (S^3, K_2)$ is an oriented knot cobordism, and W is a rational homology cobordism, then*

$$|\Upsilon_{K_2}(t) - \Upsilon_{K_1}(t)| \leq t \cdot g(\Sigma).$$

Proof. This follows immediately from Theorem 1.1. \square

Also, as another example, we recover [OSS14, Proposition 1.10]:

Corollary 12.5. *If K_- and K_+ are knots in S^3 , which differ by a crossing change, then*

$$\Upsilon_{K_+}(t) \leq \Upsilon_{K_-}(t) \leq \Upsilon_{K_+}(t) + 1 - |t-1|.$$

Proof. Suppose that K_- and K_+ are two knots which differ by a crossing change (and K_- has the negative crossing and K_+ has the positive crossing). We can construct a negative definite link cobordism (W_1, Σ_1) from (S^3, K_-) to (S^3, K_+) and also a negative definite link cobordism (W_2, Σ_2) from (S^3, K_+) to (S^3, K_-) . Each is formed by adding a 2-handle with framing -1 , around the crossing, as shown in Figure 12.1. The homology class of the surface can be read off from the intersection number of the knot on the incoming end with the Seifert disk for the -1 framed unknot.

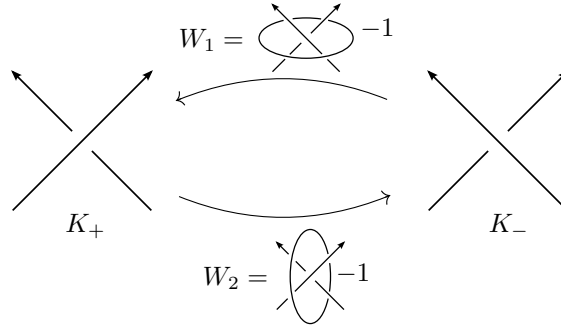


FIGURE 12.1. The two knot cobordisms (W_i, Σ_i) between (S^3, K_-) and (S^3, K_+) .

If the algebraic intersection number is n , then in $H_2(W; \mathbb{Z})$ the surface Σ induces homology class $[\Sigma] = n \cdot E$, where $E \subseteq \overline{\mathbb{CP}}^2$ is a sphere with self intersection -1 . Hence $[\Sigma_1] = 2 \cdot E$ and $[\Sigma_2] = 0$. We note that

$$M_t(0) = 0, \quad \text{and} \quad M_t(2) = -(1 - |t - 1|).$$

Applying Theorem 1.1, we see

$$\Upsilon_{K_+}(t) \leq \Upsilon_{K_-}(t) \leq \Upsilon_{K_+}(t) + 1 - |t - 1|.$$

□

12.2. Positive torus knots. We show that that the bound from Theorem 1.1 applied repeatedly to torus knots, is sharp.

According to [OSS14, Proposition 5], if n is a positive integer and $T_{n,n+1}$ denotes the $(n, n+1)$ -torus knot, then for $t \in [\frac{2i}{n}, \frac{2i+2}{n}]$, we have

$$\Upsilon_{T_{n,n+1}}(t) = -i(i+1) - \frac{1}{2}n(n-1-2i)t.$$

Using this, Feller and Krcatovich show in [FK16, Proposition 6] that $\Upsilon_K(t)$ can be computed inductively for positive torus knots, using the recursion

$$\Upsilon_{T_{a,b}}(t) = \Upsilon_{T_{a,b-a}}(t) + \Upsilon_{T_{a,a+1}}(t).$$

On the other hand, note that there is a natural knot cobordism from $(S^3, T_{a,b-a})$ to $(S^3, T_{a,b})$, obtained by -1 surgery on an unknot encircling the torus which $T_{a,b-a}$ is embedded on. If D is the Seifert disk for this unknot, we can orient D so that $T_{a,b-a}$ intersects D at a points, positively. Applying Theorem 1.1 yields

$$(17) \quad \Upsilon_{T_{a,b}}(t) \geq \Upsilon_{T_{a,b-a}}(t) + M_t(a).$$

A special case is the knot cobordism from $T_{n,1}$ (the unknot) to $T_{n,n+1}$. In this case, our bound reads

$$\Upsilon_{T_{n,n+1}}(t) \geq 0 + M_t(n) = \max_{a \in 2\mathbb{Z}+1} \frac{-a^2 + 1 + 2atn - 2tn^2}{4}.$$

On the other hand, using the computation from [OSS14] of $\Upsilon_{T_{n,n+1}}(t)$, and the following computation, we see the bound from Equation (17) is sharp:

Lemma 12.6. *If $t \in [\frac{2i}{n}, \frac{2i+2}{n}]$ then*

$$M_t(n) = -i(i+1) - \frac{1}{2}n(n-1-2i)t.$$

Proof. We claim that for t in the stated interval, the maximum of $\frac{1}{4}(-a^2 + 1 + 2tan - 2tn^2)$ occurs at $a = 2i + 1$. Write $F_a(t) = \frac{1}{4}(-a^2 + 1 + 2atan - 2tn^2)$. Note that plugging $a = 2i + 1$ into the expression for $F_a(t)$ yields the expression in the lemma statement. For real a and a fixed t , the maximum of $F_a(t)$, a parabola in a , occurs at $a = tn \in [2i, 2i+2]$. Hence for a fixed t the maximum of $F_t(a)$ over odd integers a will occur at one of $2i-1, 2i+1$, or $2i+3$. It is now an easy matter to compute that the lines $F_{2i-1}(t)$ and $F_{2i+1}(t)$, intersect at $t = 2i/n$ and the lines $F_{2i+1}(t)$ and $F_{2i+3}(t)$ intersect at $t = 2(i+1)/n$. As the slope of the line $F_a(t)$ is increasing in a , it follows that on the stated interval, $M_t(n) = F_{2i+1}(t)$, completing the proof. □

13. THE ADJUNCTION RELATION

As another application of our grading formula, we prove an adjunction relation similar to [OS04c, Theorem 3.1]. The standard adjunction inequality for $HF^+(Y, \mathfrak{s})$ follows from this relation.

If Σ is a closed surface and $A_1, \dots, A_g, B_1, \dots, B_g$ is a symplectic basis of $H_1(\Sigma; \mathbb{Z})$ which is obtained by taking a collection of simple closed curves A_i and B_i on the surface with geometric intersection numbers $|A_i \cap B_j| = \delta_{ij}$, we let $\xi(\Sigma)$ denote the element

$$\xi(\Sigma) = \prod_{i=1}^{g(\Sigma)} (U + A_i \cdot B_i) \in \mathbb{Z}_2[U] \otimes \Lambda^*(H_1(\Sigma; \mathbb{Z})).$$

The element $\xi(\Sigma)$ is defined in terms of the chosen symplectic basis.

Theorem 1.7. *Suppose that Σ is a connected, closed, oriented, embedded surface (of any genus), inside of a cobordism $W : Y_1 \rightarrow Y_2$ with W, Y_1 and Y_2 connected, and that $\mathfrak{s} \in \text{Spin}^c(W)$ is a Spin^c structure with*

$$\langle c_1(\mathfrak{s}), [\Sigma] \rangle - [\Sigma] \cdot [\Sigma] = -2g(\Sigma),$$

then

$$F_{W, \mathfrak{s}}^\circ(\cdot) = F_{W, \mathfrak{s} - PD[\Sigma]}^\circ(\iota_*(\xi(\Sigma)) \otimes \cdot),$$

for $\circ \in \{+, -, \infty\}$, where $\iota_* : H_1(\Sigma; \mathbb{Z}) \rightarrow H_1(W; \mathbb{Z})$ is the map induced by inclusion. Here $\xi(\Sigma)$ is the element described above, for a choice of symplectic basis of $H_1(\Sigma; \mathbb{Z})$.

Remark 13.1. The above theorem is phrased in terms of the $+$, $-$ and ∞ flavors, with a single basepoint, so we don't have to keep track of paths. For the hat flavor, an analogous statement holds, but one needs to keep track of a path.

Remark 13.2. When $g(\Sigma) > 0$, analogous adjunction relations can be found in [OS04c, Proposition 3.1] (for Heegaard Floer) and [OS00a, Theorem 1.3] (for Seiberg–Witten). Let us briefly compare our version to theirs, which is corrected by a factor of $\epsilon = \pm 1$ on $PD[\Sigma]$, equal to the sign of $\langle c_1(\mathfrak{s}), \Sigma \rangle$. If $\epsilon = -1$, then our formulas agree. If $\epsilon = +1$, then the surface $-\Sigma$ obtained by reversing orientation satisfies $\langle c_1(\mathfrak{s}), [-\Sigma] \rangle - [-\Sigma] \cdot [-\Sigma] \leq -2g(-\Sigma)$. Add n homologically trivial handles to the type- \mathbf{z} region of $-\Sigma$ to turn the previous inequality into an equality, and let Σ' be the resulting surface. We note that $[\Sigma'] = -[\Sigma]$. Hence our adjunction relation now reads

$$F_{W, \mathfrak{s}}^\circ(\cdot) = F_{W, \mathfrak{s} + \epsilon PD[\Sigma]}^\circ(\iota_*(\xi(\Sigma')) \otimes \cdot),$$

and now we observe that from our explicit description of ξ , we have $\iota_*(\xi(\Sigma')) = U^n \cdot \iota_*(\xi(\Sigma))$, which fits nicely with the formula from [OS00a, Theorem 1.3].

Remark 13.3. If Σ has genus zero, then the situation is similar to [FS95, Lemma 5.2]. To see that our result is consistent, suppose that S is an embedded sphere of negative self intersection with $|\langle c_1(\mathfrak{s}), S \rangle| + S \cdot S \geq 0$ (this is the situation they consider). In this case, let $\epsilon \in \{\pm 1\}$ be the sign of $\langle c_1(\mathfrak{s}), S \rangle$. We have

$$\langle c_1(\mathfrak{s}), -\epsilon S \rangle - S \cdot S \leq -2g(S).$$

Adding n null-homologous handles to the surface to achieve equality in the previous equation, and then applying our result, we see that

$$F_{W, \mathfrak{s}}^\circ = U^n \cdot F_{W, \mathfrak{s} + \epsilon PD[S]}^\circ.$$

If S is an embedded sphere in a 4-manifold X which admits an admissible cut which is disjoint from S , we recover an analog of [FS95, Lemma 5.2], for the Ozsváth–Szabó mixed invariant.

To prove the above theorem, we will consider what happens to the maps $F_{W, F, \mathfrak{s}}^-$ when we set either $U = 1$ or $V = 1$. To find out what the maps are, we need two results from [Zem16b]. The first is [Zem16b, Proposition 7.15], which states that if the ends of B are adjacent to basepoints w_1, w_2, z_1 and z_2 , then

$$(18) \quad F_B^{\mathbf{z}} \simeq \Phi_{w_1} F_B^{\mathbf{w}} + F_B^{\mathbf{w}} \Phi_{w_1}.$$

(Note also that the equivalence is easily seen to also hold if we replace both instances of w_1 with w_2). The second relation we need is [Zem16b, Lemma 14.1], which implies that after we set all $U_{\mathbf{w}}$ variables equal to 1, if λ is the arc of \mathbb{L} going from z to z' , containing only the basepoint w , then we have

$$(19) \quad \Phi_w|_{U_{\mathbf{w}}=1} \simeq A_\lambda + V_z \Psi_z.$$

We note here we are writing Ψ_z for the map on CF^- that we would normally write Φ_z for, using the notation from [Zem15b]. In the case that the link component containing w has only two basepoints, the relative homology map A_λ is replaced with the map for the normal homology action for that link component.

We use the previous two relations to prove the following lemma:

Lemma 13.4. *Suppose that $\mathbb{L} = (L, \mathbf{w}, \mathbf{z})$ is a link in Y and B_1 and B_2 are two bands attached to L . Further assume that both ends of B_1 are attached to the same component of $L \setminus (\mathbf{w} \cup \mathbf{z})$, and that the ends of B_2 are both attached to opposite sides of B_1 (see Figure 13.2). Write (W, F) for the decorated link cobordism where $W = Y \times [0, 1]$ and F is formed by adding the bands B_1 and B_2 , to \mathbb{L} , both as type- \mathbf{z} bands. Let γ_1 and γ_2 two simple closed curves on Σ be defined by taking γ_1 to be the core of B_1 concatenated with an arc*

λ_0 of $K \setminus (\mathbf{w} \cup \mathbf{z})$ between the two ends of B_1 , and define γ_2 to be the core of B_2 , as in Figure 13.2. After tensoring to set $U = 1$, the map $F_{W,F,\mathfrak{s}}^-$ on $CF^-(Y, z, \mathfrak{s} - PD[L])$ is equal to

$$(V + A_{\gamma_2} A_{\gamma_1}).$$

Proof. The extra components of L play no role in the proof, so let us assume for the sake of simplicity that $\mathbb{L} = \mathbb{K} = (K, \mathbf{w}, \mathbf{z})$, for a knot K . Also for notational simplicity, we will assume that \mathbb{K} has exactly two basepoints, w_1 and z_1 . For knots with more basepoints, the assumption that B_1 has ends in the same component of $K \setminus (\mathbf{w} \cup \mathbf{z})$ will be needed, though the proof will be essentially unchanged.

The proof is to decompose the surface nicely, and then use relations in the previous paragraphs from [Zem16b]. We consider the decomposition shown in Figure 13.1.

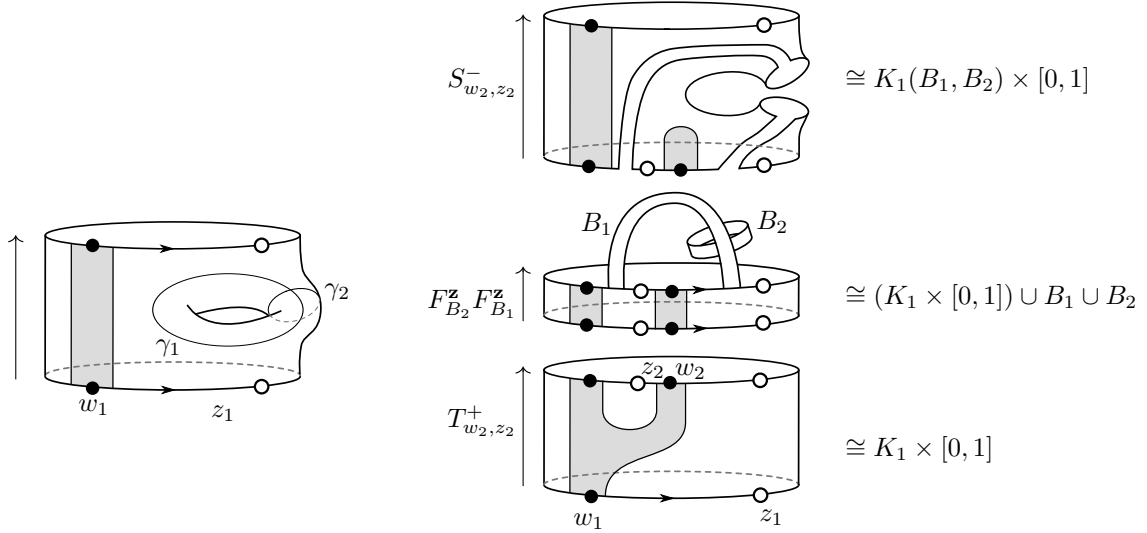


FIGURE 13.1. A decomposition of the surface with divides F into three pieces which we can use to compute the map. The first piece corresponds to a T_{w_2, z_2}^+ quasi-stabilization. The second piece is two type- \mathbf{z} bands, and the third piece is an S_{w_2, z_2}^- quasi-destabilization.

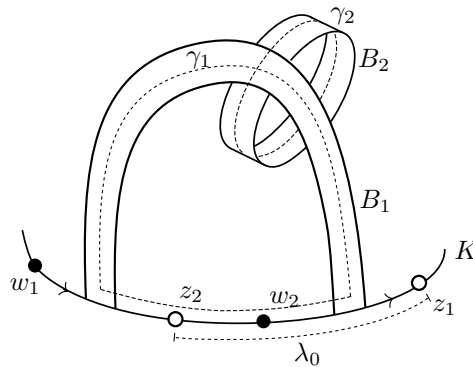


FIGURE 13.2. The homology classes γ_1 and γ_2 , as well as the relative homology class λ_0 .

Using this decomposition, the maps are

$$F_{W,F,\mathfrak{s}}^\circ \simeq S_{w_2, z_2}^- F_{B_2}^z F_{B_1}^z T_{w_2, z_2}^+.$$

Note that by direct computation of the quasi-stabilized differential in [Zem16b], we can compute that $S_{w_2, z_2}^- \simeq T_{w_2, z_2}^- \Phi_{w_2}$. Thus by using Equations (18) and (19), we see that $F_{W, F, \mathfrak{s}}^\circ$ is equivalent to

$$T_{w_2, z_2}^- \Phi_{w_2} (\Phi_{w_2} F_{B_2}^\mathbf{w} + F_{B_2}^\mathbf{w} \Phi_{w_2}) (\Phi_{w_2} F_{B_1}^\mathbf{w} + F_{B_1}^\mathbf{w} \Phi_{w_2}) T_{w_2, z_2}^+$$

which is equivalent to

$$T_{w_2, z_2}^- \Phi_{w_2} F_{B_2}^\mathbf{w} \Phi_{w_2} F_{B_1}^\mathbf{w} \Phi_{w_2} T_{w_2, z_2}^+$$

since $\Phi_{w_2}^2 \simeq 0$.

Note that when we set $U = 1$, the map Ψ_z (on CFL^-) becomes Ψ_z (on CF^-).¹ Similarly, the quasi-stabilization map T_{w_2, z_2}^\pm becomes the free stabilization map $S_{z_2}^\pm$, once we set $U = 1$. Hence, after tensoring with $\mathbb{Z}_2[U, V]/(U - 1)$, and using Equation (19), we get that this is now equivalent to

$$S_{z_2}^-(A_{\lambda_0} + A_{\gamma_2} + V\Psi_{z_2})(A_{\gamma_1} + V\Psi_{z_2})(A_{\lambda_0} + V\Psi_{z_2})S_{z_2}^+,$$

as maps on CF^- (note that after tensoring with that module, T_{w_2, z_2}^\pm becomes $S_{z_2}^\pm$). Note also that $\Psi_{z_2} S_{z_2}^+ \simeq 0$ and $S_{z_2}^- \Psi_{z_2} \simeq 0$, as $\Psi_{z_2} \simeq S_{z_2}^+ S_{z_2}^-$ and $S_{z_2}^- S_{z_2}^+ \simeq 0$. Hence the above expression can be reduced to

$$S_{z_2}^-(A_{\lambda_0} + A_{\gamma_2})(A_{\gamma_1} + V\Psi_{z_2})(A_{\lambda_0})S_{z_2}^+,$$

which becomes

$$S_{z_2}^- A_{\lambda_0} A_{\gamma_1} A_{\lambda_0} S_{z_2}^+ + S_{z_2}^- A_{\gamma_2} V\Psi_{z_2} A_{\lambda_0} S_{z_2}^+ + S_{z_2}^- A_{\lambda_0} V\Psi_{z_2} A_{\lambda_0} S_{z_2}^+ + S_{z_2}^- A_{\gamma_2} A_{\gamma_1} A_{\lambda_0} S_{z_2}^+.$$

We now manipulate each term individually, using the relations of the homology actions, the relative homology maps, and the free-stabilization maps derived in [Zem15b]. We compute as follows:

$$S_{z_2}^- A_{\lambda_0} A_{\gamma_1} A_{\lambda_0} S_{z_2}^+ \simeq S_{z_2}^- A_{\lambda_0}^2 S_{z_2}^- A_{\gamma_1} \simeq S_{z_2}^- V S_{z_2}^+ A_{\gamma_1} \simeq 0;$$

$$S_{z_2}^- A_{\gamma_2} V\Psi_{z_2} A_{\lambda_0} S_{z_2}^+ \simeq A_{\gamma_2} V S_{z_2}^- \Psi_{z_2} A_{\lambda_0} S_{z_2}^+ \simeq 0;$$

$$S_{z_2}^- A_{\lambda_0} V\Psi_{z_2} A_{\lambda_0} S_{z_2}^+ \simeq V S_{z_2}^- \Psi_{z_2} A_{\lambda_0} A_{\lambda_0} S_{z_2}^+ + V S_{z_2}^- A_{\lambda_0} S_{z_2}^+ \simeq V;$$

and

$$S_{z_2}^- A_{\gamma_2} A_{\gamma_1} A_{\lambda_0} S_{z_2}^+ \simeq A_{\gamma_2} A_{\gamma_1} S_{z_2}^- A_{\lambda_0} S_{z_2}^+ \simeq A_{\gamma_2} A_{\gamma_1}.$$

Adding up the above contributions yields $V + A_{\gamma_2} A_{\gamma_1}$, completing the proof. \square

Proof of Theorem 1.7. Push Σ so that a disk D_1 of Σ lies in Y_1 and a disk D_2 of Σ lies in Y_2 . We then form the decorated link cobordism $(W, F) = (W, \Sigma \setminus (B_1 \cup B_2), \mathcal{A})$, where \mathcal{A} are two parallel arcs on Σ running from $\mathbb{U}_1 = \partial B_1$ to $\mathbb{U}_2 = \partial B_2$. We set the region between the two parallel arcs to be $\Sigma_{\mathbf{w}}$, and we set the rest to be $\Sigma_{\mathbf{z}}$. We will consider the link cobordism maps $F_{W, F, \mathfrak{s}}^-$.

Using the disks D_1 and D_2 for Seifert surfaces of \mathbb{U}_1 and \mathbb{U}_2 , the induced Alexander grading due to the map $F_{W, F, \mathfrak{s}}^-$ on link Floer homology is zero, since

$$\frac{\langle c_1(\mathfrak{s}), \Sigma \rangle - [\Sigma] \cdot [\Sigma]}{2} + g(\Sigma) = 0,$$

by assumption. We pick diagrams for Y_i so that the basepoints w and z are in the same region on the Heegaard surface, and the Seifert disks D_i intersect the Heegaard surface in an arc between the two basepoints, which is also contained in the same region as the basepoints. We note that $CFL^-(Y_i, \mathbb{U}_i, \mathfrak{s}|_{Y_i})$ is generated by monomials $U^m V^n \cdot \mathbf{x}$ with $m, n \geq 0$. The subcomplex in zero Alexander grading is generated by monomials with $m = n$. The fact that the Alexander grading change is zero implies that the zero graded portion of $CFL^-(Y_1, \mathbb{U}_1, \mathfrak{s}|_{Y_1})$ is mapped to the zero graded portion of $CFL^-(Y_2, \mathbb{U}_2, \mathfrak{s}|_{Y_1})$. Note that there are two ways to identify $CFL^-(Y_i, \mathbb{U}_i, \mathfrak{s}|_{Y_i})$ with $CF^-(Y_i, \mathfrak{s}|_{Y_i})$: we can set $U = 1$ or we can set $V = 1$. Let U_0 be a formal variable, which we will use for clarity. Since the Alexander grading change is zero, the map induced by setting $U = 1$ and $V = U_0$ is equal to the map induced by setting $U = U_0$ and $V = 1$. Hence our strategy is thus to show that the $V = 1$ map can be naturally identified with $F_{W, \mathfrak{s}}^-(\cdot)$, and the $U = 1$ map can be naturally identified with $F_{W, \mathfrak{s} - PD[\Sigma]}^-(\iota_*(\xi(\Sigma)) \otimes \cdot)$.

Take a parametrized Kirby decomposition of (W, F) (see [Zem16b, Definition 9.5]) obtained from a Morse function f with gradient like vector field v (in the sense of [Zem16b, Definition 10.7]) and a collection of regular values \mathbf{b} . Write $\mathcal{K} = \mathcal{K}_n \circ \dots \circ \mathcal{K}_1$ where each \mathcal{K}_i is an elementary parametrized link cobordism. That

¹Using the notation from [Zem15b], we would normally write Φ_z for this map on CF^- , but we will persist in writing Ψ_z .

is, each \mathcal{K}_i either corresponds to a single 1-handle or 3-handle away from the link, a collection of 2-handles away from the link, a single critical point of the divides, or a 4-dimensional index 1 critical point which occurs along Σ . We arrange so that the index 1 critical points along Σ and critical points of the divides occur first, then afterwards come the index 1 critical points away from Σ , then the index 2 critical points, and then finally the index 3 critical points. The maps for index 1 critical points along Σ are a composition of a 1-handle map, a band map, and some diffeomorphism maps.

The 1-handle maps commute with the band maps by the triangle map computation of [Zem15b, Theorem 8.8], and hence we can pull all of the 1-handles down, to write the link cobordism map as a composition of 1-handle maps, then band maps, then a single 2-handle map (adding all the 2-handles at once), finally followed by some 3-handle maps. Write $W = W_3 \cup W_2 \cup W_S \cup W_1$, where W_i is formed by adding i -handles away from the link, and W_S is diffeomorphic to $Y \times [0, 1]$ with a surface formed by adding bands inside of Y , and the dividing set on W_S contains all of the critical points of the divides. Write $F_{W,F,\mathfrak{s}}|_{U=1}$ for the map $F_{W,F,\mathfrak{s}}^-$ after we set $U = 1$ and $V = U_0$ (for a formal variable U_0), and set $F_{W,F,\mathfrak{s}}|_{V=1}$ to be the map obtained by setting $V = 1$ and $U = U_0$. Write (W_i, F_i) for $(W_i, F \cap W_i)$ and write \mathfrak{s}_i for $\mathfrak{s}|_{W_i}$, and define $F_S \subseteq W_S$ and \mathfrak{s}_S similarly. Write Σ_i for the underlying surface of F_i .

By the composition law, we have

$$(20) \quad F_{W,F,\mathfrak{s}}^- = F_{W_3,F_3,\mathfrak{s}_3}^- \circ F_{W_2,F_2,\mathfrak{s}_2}^- \circ F_{W_S,F_S,\mathfrak{s}_S}^- \circ F_{W_1,F_1,\mathfrak{s}_1}^-.$$

We first note that as any \mathbf{z} -band map turns into a change of diagrams map once we set $V = 1$, as we use the top $\mathbf{gr}_{\mathbf{w}}$ -graded intersection point for the \mathbf{z} -band maps, which is the same intersection point we use for the change of diagrams map once we forget about the \mathbf{z} -basepoints. Hence, as in [Zem16b, Section 14], since $\Sigma_{\mathbf{w}}$ is just a strip, the map $F_{W_S,F_S,\mathfrak{s}_S}^-|_{V=1}$ becomes the graph cobordism map for a graph which is equal to a path with some trivial strands added. Since we can always remove the trivial strands by basic relations of the graph cobordism maps, once we set $V = 1$ the map $F_{W_S,F_S,\mathfrak{s}_S}^-$ is just the identity map. Upon direct inspection, we have that

$$F_{W_i,F_i,\mathfrak{s}_i}^-|_{V=1} = F_{W_i,\mathfrak{s}_i}^-$$

for the handle attachment cobordisms (W_i, F_i) . Using the composition law again, we see that $F_{W,F,\mathfrak{s}}^-|_{V=1} = F_{W,\mathfrak{s}}^-$.

We now consider the map $F_{W,F,\mathfrak{s}}^-|_{U=1}$. Note that the 1-handle maps and 3-handle maps in the composition for $F_{W,F,\mathfrak{s}}^-|_{U=1}$ agree with the maps for $F_{W,\mathfrak{s}-PD[\Sigma]}^-$. Similarly, noting that $(\mathfrak{s} - PD[\Sigma])|_{W_S} = \mathfrak{s}_S$, we have

$$F_{W_S,F_S,(\mathfrak{s}-PD[\Sigma])|_{W_S}}^-|_{U=1} = F_{W_S,F_S,\mathfrak{s}_S}^-|_{U=1} = \iota_* \prod_{i=1}^{g(\Sigma)} (V + A_i \cdot B_i),$$

by Lemma 13.4 and the composition law.

We finally note that the triangle maps used in the construction of the link cobordism maps in [Zem16b] are not entirely symmetric between the \mathbf{w} - and the \mathbf{z} -basepoints. The triangle maps involved in 2-handle maps count triangles which satisfy $\mathfrak{s}_{\mathbf{w}}(\psi) = \mathfrak{s}$. The band maps also count holomorphic triangles, but since all of the links resulting from taking \mathbb{U}_1 and adding bands are null-homologous (in the sense that the total class is null-homologous), the maps $\mathfrak{s}_{\mathbf{w}}$ and $\mathfrak{s}_{\mathbf{z}}$ appearing in the band maps coincide. For the 2-handle maps, the maps $\mathfrak{s}_{\mathbf{w}}$ and $\mathfrak{s}_{\mathbf{z}}$ may be different. The map F_{W_2,\mathfrak{s}_2} counts triangles with $\mathfrak{s}_{\mathbf{w}}(\psi) = \mathfrak{s}_2$, but by Lemma 3.3, these are exactly the same triangles which satisfy $\mathfrak{s}_{\mathbf{z}}(\psi) = \mathfrak{s}_2 - PD[\Sigma_2]$. These are the triangles counted by $F_{W_2,\mathfrak{s}_2-PD[\Sigma_2]}$. Hence we conclude that $F_{W,F,\mathfrak{s}}^-|_{U=1}$ is equal to

$$F_{W,F,\mathfrak{s}}^-|_{U=1} = F_{W_3,\mathfrak{s}_3}^- \circ F_{W_2,\mathfrak{s}_2-PD[\Sigma_2]}^- \circ [\iota_*(\xi(\Sigma))] \circ F_{W_1,\mathfrak{s}_1}^-.$$

Hence, using the composition law, we see that $F_{W,F,\mathfrak{s}}^-|_{U=1}$ is the map

$$F_{W,\mathfrak{s}-PD[\Sigma]}^-(\iota_*(\xi(\Sigma)) \otimes \cdot),$$

completing the proof. \square

More generally, one could put more exotic sets of divides on Σ , such as those considered in Theorem 1.8, and would presumably recover relations of the Heegaard Floer cobordism maps analogous to those from [OS00b] for the Seiberg-Witten invariant.

We now use the adjunction relation above to give a link Floer homology proof of the standard adjunction inequality, proven by Ozsváth and Szabó in [OS04a].

Corollary 13.5. *If Σ is a closed, oriented surface in Y with $g(\Sigma) > 0$ and $HF^+(Y, \mathfrak{s}) \neq 0$, then*

$$|\langle c_1(\mathfrak{s}), \Sigma \rangle| \leq 2g(\Sigma) - 2.$$

Proof. We apply the previous adjunction relation to the identity cobordism $W = Y \times [0, 1]$. If Σ is a closed, oriented surface in Y which violates the inequality, we reverse the orientation of Σ if necessary, and add null-homologous handles so that

$$(21) \quad \langle c_1(\mathfrak{s}), \Sigma \rangle = -2g(\Sigma)$$

(note that $\langle c_1(\mathfrak{s}), \Sigma \rangle$ is always even). Applying the previous theorem to the identity cobordism $Y \times [0, 1]$, we see that

$$\text{id} = [\iota_*(\xi(\Sigma))] \cdot F_{Y \times [0, 1], \mathfrak{s} - PD[\Sigma]}^+.$$

As $[\Sigma] = 0 \in H_2(W, \partial W; \mathbb{Z}) \cong H^2(W; \mathbb{Z})$, we have that $F_{W, \mathfrak{s} - PD[\Sigma]}^+ = F_{W, \mathfrak{s}}^+ = \text{id}$. Hence

$$\text{id} = [\iota_*(\xi(\Sigma))].$$

On the other hand, \mathfrak{s} must be non-torsion for Equation (21) to be satisfied. As a consequence, we have that $U^n \cdot HF^+(Y, \mathfrak{s}) = 0$ for sufficiently large n , by [OS04c, Lemma 2.3]. Similarly $[\gamma] \cdot [\gamma] = 0$ for any $[\gamma] \in H_1(Y; \mathbb{Z})$. As such, if n is sufficiently large, the action of $[\iota_*(\xi(\Sigma))]^n$ will be zero on $HF^+(Y, \mathfrak{s})$. Hence $\text{id} = \text{id}^n = [\iota_*(\xi(\Sigma))]^n = 0$ as maps on $HF^+(Y, \mathfrak{s})$, so $HF^+(Y, \mathfrak{s})$ must vanish. \square

As in [OS06, Theorem 1.5], the adjunction inequality for the Ozsváth–Szabó mixed invariant for surfaces with nonnegative self intersection inside of a 4-manifold follows from the above adjunction inequality by using the blowup formula to reduce to the case that $[\Sigma] \cdot [\Sigma] = 0$, and factoring the cobordism map on HF^+ through a regular neighborhood of Σ .

14. THE MAPS ON \mathcal{HFL}^∞ FOR SURFACES IN NEGATIVE DEFINITE 4-MANIFOLDS

In this section we compute the maps associated to knot cobordisms on homology when the 4-manifold is negative definite and the dividing set is relatively simple.

Theorem 1.8. *Suppose that $(W, F) : (S^3, \mathbb{K}_1) \rightarrow (S^3, \mathbb{K}_2)$ is a knot cobordism such that $b_1(W) = b_2^+(W) = 0$, $F = (\Sigma, \mathcal{A})$ is a connected surface, and \mathbb{K}_i are two null-homologous knots, each with two basepoints. Suppose further that \mathcal{A} consists of two arcs going from K_1 to K_2 and $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$ are both connected. Then the induced map on homology*

$$F_{W, F, \mathfrak{s}}^\infty : \mathcal{HFL}^\infty(S^3, \mathbb{K}_1) \rightarrow \mathcal{HFL}^\infty(S^3, \mathbb{K}_2)$$

is an isomorphism. In fact, under the identification $\mathcal{HFL}^\infty(S^3, \mathbb{K}_i) \cong \mathbb{Z}_2[U, V, U^{-1}, V^{-1}]$, it is the map

$$1 \mapsto U^{-d_1/2} V^{-d_2/2},$$

where

$$d_1 = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4} - 2g(\Sigma_{\mathbf{w}})$$

and

$$d_2 = \frac{c_1(\mathfrak{s} - PD[\Sigma])^2 - 2\chi(W) - 3\sigma(W)}{4} - 2g(\Sigma_{\mathbf{z}}).$$

Proof. Our absolute gradings give a canonical identification of $\mathcal{HFL}^\infty(S^3, \mathbb{K}_i)$ with $\mathbb{Z}_2[U, V, U^{-1}, V^{-1}]$. Using our grading formulas, the map $F_{W, F, \mathfrak{s}}^\infty : \mathbb{Z}_2[U, V, U^{-1}, V^{-1}] \rightarrow \mathbb{Z}_2[U, V, U^{-1}, V^{-1}]$ is equal to $1 \mapsto c \cdot U^{-d_1/2} V^{-d_2/2}$ for some $c \in \mathbb{Z}_2$. We just need to argue that $c = 1$. For this, we note that when we set $V = 1$, the map goes from $CF^\infty(S^3)$ to $CF^\infty(S^3)$. We can apply Lemma 13.4 repeatedly, as well as the composition law, as in the proof of Theorem 1.7 to see that the map induced by $F_{W, F, \mathfrak{s}}^\infty$ on $CF^\infty(S^3)$ is equal to $F_{W, \mathfrak{s}}^\infty(\iota_*(\xi(\Sigma_{\mathbf{w}})) \otimes \cdot)$, where $\xi(\Sigma_{\mathbf{w}})$ is defined by picking an appropriate symplectic basis $A_1, B_1, \dots, A_{g(\Sigma_{\mathbf{w}})}, B_{g(\Sigma_{\mathbf{w}})}$ of $H_1(\Sigma_{\mathbf{w}}; \mathbb{Z})$ and defining $\xi(\Sigma_{\mathbf{w}}) = \prod_{i=1}^{g(\Sigma_{\mathbf{w}})} (U + A_i \cdot B_i)$. However we assumed that $b_1(W) = 0$, so the actions of A_i and B_i vanish, and we see that $F_{W, F, \mathfrak{s}}^\infty$ simply induces the map

$U^{g(\Sigma_{\mathbf{w}})} \cdot F_{W,s}^\infty$. The map $F_{W,s}^\infty$ is an isomorphism once again by the argument in the proof of [OS03a, Theorem 9.1]. In particular, we conclude that $c = 1$ so $F_{W,F,s}^\infty$ is an isomorphism on homology. \square

Remark 14.1. We can use the above strategy to compute the maps on CF^- induced by the knot cobordism maps for more general knot cobordisms. As we've seen, there are two ways to obtain a map on CF^- from the link cobordism maps $F_{W,F,s}^\infty$. The first is obtained by setting $V = 1$ and only paying attention to the \mathbf{w} -basepoints. The second is obtained by setting $U = 1$, and paying attention to only the \mathbf{z} -basepoints. If $(W, F) : (Y_1, \mathbb{K}_1) \rightarrow (Y_2, \mathbb{K}_2)$ is a knot cobordism with \mathbb{K}_i two knots each with exactly two basepoints, with $F = (\Sigma, \mathcal{A})$ a connected surface with divides, such that \mathcal{A} consists of exactly two arcs from \mathbb{K}_1 to \mathbb{K}_2 which divide Σ into two connected subsurfaces $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$ meeting along \mathcal{A} , a straightforward adaptation of the proofs of Theorem 1.7 and Theorem 1.8 shows that the $V = 1$ map is exactly $F_{W,s}^\infty(\iota_*(\xi(\Sigma_{\mathbf{w}})) \otimes \cdot)$ and the $U = 1$ map is exactly $F_{W,s-PD[\Sigma]}^\infty(\iota_*(\xi(\Sigma_{\mathbf{z}})) \otimes \cdot)$.

Corollary 1.9. *Suppose that $\Sigma \subseteq S^4$ is a closed, oriented surface, and \mathcal{A} is a simple closed curve on Σ which divides Σ into two connected subsurfaces, $\Sigma_{\mathbf{w}}$ and $\Sigma_{\mathbf{z}}$, then the link cobordism map*

$$F_{S^4,F,s_0}^- : CFL^-(\emptyset, \emptyset) \rightarrow CFL^-(\emptyset, \emptyset)$$

is the map

$$1 \mapsto U^{g(\Sigma_{\mathbf{w}})} V^{g(\Sigma_{\mathbf{z}})}.$$

In particular the maps for 2-knots in S^4 (or any homotopy S^4) are the identity.

We can apply this to compute the effect on the link cobordism maps of taking the internal connected sum of a surface Σ with a closed surface Σ_0 in a 4-ball not intersecting Σ .

Corollary 14.2. *Suppose that $(W, F) : (Y_1, \mathbb{L}_1) \rightarrow (Y_2, \mathbb{L}_2)$ is a link cobordism with $F = (\Sigma, \mathcal{A})$ and suppose there is a 4-ball $B \subseteq W$ which doesn't intersect Σ , and that $\Sigma_0 \subseteq B$ is a closed surface. Let $F' = (\Sigma \# \Sigma_0, \mathcal{A})$ denote the internal connected sum, taken at a point in $\Sigma_{\mathbf{z}} \subseteq \Sigma$, and the entire surface Σ_0 is given the designation of type \mathbf{z} . If t denotes the color of Σ_0 , then*

$$F_{W,F',s}^- = V_t^{g(\Sigma_0)} \cdot F_{W,F,s}^-.$$

Proof. The key idea is that the cobordism maps from [Zem16b] allow us to puncture the 4-manifold and surface, creating a new end, which we can push the surface Σ_0 into.

Let B' be a 4-ball containing Σ_0 and intersecting Σ in a disk along an arc of \mathcal{A} , and assume that the internal connected sum is taken inside of the ball B' . Define $\mathcal{B}_1 = (B', F_1)$ to be the link cobordism from \emptyset to (S^3, \mathbb{U}) (here \mathbb{U} is the unknot in S^3 with exactly two basepoints), formed by setting $F_1 = (\Sigma_0 \# \Sigma) \cap B'$. Define $F_2 = \Sigma \cap B' \subseteq B'$, and let $\mathcal{B}_2 = (B', F_2)$. Let

$$(W_0, F_0) = (W \setminus B', F \setminus B'),$$

viewed as a cobordism from $(Y_1 \sqcup S^3, \mathbb{L}_1 \sqcup \mathbb{U})$ to (Y_2, \mathbb{L}_2) . Applying Corollary 1.9 and [Zem16b, Lemma 12.4], we have

$$\begin{aligned} F_{W,F',s}^- &= F_{W_0,F_0,s_0}^- \circ F_{\mathcal{B}_1,s_0}^- \\ &= F_{W_0,F_0,s_0}^- \circ (V_t^{g(\Sigma_0)} \cdot F_{\mathcal{B}_2,s_0}^-) \\ &= V_t^{g(\Sigma_0)} \cdot F_{W_0,F_0,s_0}^- \circ F_{\mathcal{B}_2,s_0}^- \\ &= V_t^{g(\Sigma_0)} \cdot F_{W,F,s}^-. \end{aligned}$$

\square

A natural question in the spirit of the above computations is the following:

Question 14.3. Can the link cobordism maps differentiate two slice disks of a given knot?

Using the previous corollary, we see that if D_1 and D_2 are two slice disks for a knot \mathbb{K} such that there are 2-knots S_1 and S_2 , such that S_i is contained in a 4-ball not intersecting D_i such that $D_1 \# S_1$ and $D_2 \# S_2$ are isotopic, then the maps from $\mathbb{Z}_2[U, V] = CFL^-(S^3, U, w, z)$ to $CFL^-(S^3, \mathbb{K})$ associated to D_1 and D_2 agree.

REFERENCES

- [FK16] P. Feller and D. Krcatovich, *On cobordisms between knots, braid index, and the Upsilon-invariant* (2016), arXiv:1602.02637v1[math.GT].
- [FR79] R. Fenn and C. Rourke, *On Kirby's calculus of links*, *Topology* **18** (1979), 1–15.
- [FS95] R. Fintushel and R. Stern, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, *Turkish J. Math* **19** (1995), 145–157.
- [HKM08] K. Honda, W. Kazez, and G. Matić, *Contact structures, sutured Floer homology and TQFT* (2008), arXiv:0807.2431 [math.GT].
- [HMZ] K. Hendricks, C. Manolescu, and I. Zemke, *A connected sum formula for involutive Heegaard Floer homology*, arxiv:1607.07499[math.GT].
- [Juh16] A. Juhász, *Cobordisms of sutured manifolds and the functoriality of link Floer homology*, *Adv. in Math.* **299** (2016), 940–1038.
- [JM16a] A. Juhász and M. Marengon, *Concordance maps in knot Floer homology*, *Geom. Topol.* **20** (2016), 3623–3673.
- [JM16b] ———, *Computing cobordism maps in link Floer homology and the reduced Khovanov TQFT* (2016), arXiv:1503.00665[math.GT].
- [JT12] A. Juhász and D. P. Thurston, *Naturality and mapping class groups in Heegaard Floer homology* (2012), arXiv:1210.4996 [math.GT].
- [Kir78] R. Kirby, *A calculus for framed links in S^3* , *Invent. Math* **45** (1978), 35–56.
- [Lip06] R. Lipshitz, *A cylindrical reformulation of Heegaard Floer homology*, *Geom. Topol.* **10** (2006), 955–1097.
- [GS99] R. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, American Mathematical Society, Providence, RI, 1999.
- [HM15] K. Hendricks and C. Manolescu, *Involutive Heegaard Floer homology*, arXiv:1507.00383[math.GT] (2015).
- [MO10] C. Manolescu and P. Ozsváth, *Heegaard Floer homology and integer surgeries on links* (2010), arXiv:1011.1317v3 [math.GT].
- [OS00a] P. S. Ozsváth and Z. Szabó, *The symplectic Thom conjecture*, *Ann. of Math* **151** (2000), 93–124.
- [OS00b] ———, *Higher type adjunction inequalities in Seiberg–Witten theory*, *J. Differential Geom.* **55** (2000), no. 3, 385–440.
- [OS03a] ———, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, *Adv. in Math.* **173** (2003), no. 2, 179–261.
- [OS03b] ———, *Knot Floer homology and the four-ball genus*, *Geom. Topol.* **7** (2003), 615–639.
- [OS04a] ———, *Holomorphic disks and three-manifold invariants: properties and applications*, *Ann. of Math (2)* **159** (2004), no. 3, 1159–1245.
- [OS04b] ———, *Holomorphic disks and topological invariants for closed three manifolds*, *Ann. of Math. (2)* **159** (2004), no. 3, 1027–1158.
- [OS04c] ———, *Holomorphic triangle invariants and the topology of symplectic four-manifolds*, *Duke Math J.* **121** (2004), no. 1, 1–34.
- [OS04d] ———, *Holomorphic disks and knot invariants*, *Adv. Math* **186** (2004), 58–116.
- [OS06] ———, *Holomorphic triangles and invariants for smooth four-manifolds*, *Adv. in Math.* **202**, (2006), Issue 2, 326–400.
- [OS08] ———, *Holomorphic disks, link invariants, and the multi-variable Alexander polynomial*, *Alg. & Geom. Top.* **8** (2008), 615–692.
- [OSS14] P. Ozsváth, A. Stipsicz, and Z. Szabó, *Concordance homomorphisms from knot Floer homology* (2014), arXiv:1407.1795v2[math.GT].
- [Ras03] J. Rasmussen, *Floer homology and knot complements* (2003), arXiv:0306378 [math.GT].
- [Rob97] Justin Roberts, *Kirby calculus in manifolds with boundary*, *Turkish J. Math* **21**, (1997), no. 1, 111–117.
- [Sar11a] S. Sarkar, *Maslov index formulas for Whitney n -gons*, *J. of Symp. Geom.* **9** (2011), Number 2, 251–270.
- [Sar11b] ———, *Grid diagrams and the Ozsváth–Szabó tau-invariant*, *Math. Res. Let.* **18** (2011), Number 6, 1239–1257.
- [Zem15a] I. Zemke, *A graph TQFT for hat Heegaard Floer homology* (2015), arXiv:1010.2808 [math.GT].
- [Zem15b] ———, *Graph cobordisms and Heegaard Floer homology* (2015), arXiv:1512.01184 [math.GT].
- [Zem16a] ———, *Quasi-stabilization and basepoint moving maps in link Floer homology* (2016), arxiv:1604.04316 [math.GT].
- [Zem16b] ———, *Link cobordisms and functoriality in link Floer homology* (2016), arXiv:1610.05207 [math.GT].

DEPARTMENT OF MATHEMATICS, UCLA, 520 PORTOLA PLAZA, LOS ANGELES, CA 90095, USA
E-mail address: ianzenke@math.ucla.edu